# DYNAMIC RIGHT COPRIMT FACTORIZATION AND OBSERVER DESIGN FOR NONLINEAR SYSTEMS*

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***Abstract*— The output behavior of a nonlinear control system depends not only on its input but also on its initial conditions. These two factors have to be considered simultaneously in nonlinear systems design. This paper presents a new definition, called dynamic right factorization, for nonlinear dynamic control systems. This factorization takes the initial conditions as well as the system input into account. Coprimeness and fundamental properties of dynamic right coprime factorization are investigated. Its relations to the system observability and the observer design problem are discussed. An example is given to illustrate the procedure of obtaining the dynamic right factorization and designing an observer for a given nonlinear dynamic system.**

***Keywords* — nonlinear system; initial condition; right coprime factorization; observer.**

## I. INTRODUCTION

In the operator-theoretic approach to the study of control systems, a system Σ is considered as a mapping *P* from its input space *U* to its output space *Y*, i.e., $\Sigma P{:}U \to Y$. $P$ is called the input-output mapping of Σ. Since most control systems are dynamic systems whose output behavior depends not only on its inputs but also on its states, thereby relying on the initial conditions of the systems. Precisely, the mapping of a dynamic system should be defined from a Cartesian space $X_0 \times U$ to $Y$, i.e.,

$$\Sigma \quad \begin{array}{l} P{:}X_0 \times U \to Y \\ (x_0, u) \mapsto y \end{array} \tag{1.1}$$

where $X_0$ is the linear space of initial states associated with the system.

Initial states are considered, even in linear control systems, and are manipulated independently of system inputs. For linear control systems, by the linearity, we have

$$P(x_0,u) = P(x_0,0) + P(0,u) = T(x_0) + G(u)\,, \tag{1.2}$$

where the operators $T$ and $G$ are defined by $T{:}X_0 \to Y$; $T(x_0) = P(x_0,0)$ and $G{:}U \to Y$;. $G(u) = P(0,u)$. Equation (1.2) implies that the effects of $X_0$ and $U$ can be separately considered.

For a nonlinear control system, however, this separation does not hold in general. In the nonlinear case, more often than not, the initial state and the input have to be considered simultaneously. It is inappropriate to fix the initial state in the design and analysis for a nonlinear control system, since the dynamic behavior of a nonlinear system strongly depends on its initial conditions. For example, the systems may be stable for initial states within a set of the initial space $X_0$, but unstable elsewhere.

In the operator-theoretic approach to control systems, coprime factorization is one of the existing efficient methods for analysis and design (Wolovich, 1974; Kailath, 1980; Vidyasagar, 1985; Youla *et al.*, 1976). By applying the operator factorization methodology, one can introduce the so-called quasi-state space, a framework similar to the state space for linear systems, for nonlinear control systems.

Nevertheless, since 1980's, the mathematical nonlinear operator theory has been introduced to design, analysis, stabilization and optimization of nonlinear control systems (see, for example, (Banos, 1994; Chen & Han, 1998; Chen & Figueiredo, 1992; Desoer & Kabuli, 1988; Hammer, 1987, 1994; Han & Rao, 1995; Paice, *et al.*, 1993; Sontag, 1989; Verma & Hunt, 1993) and the references therein). However, most papers only consider the operator $\Sigma P{:}U \to Y$. The initial condition seems to be left out so that these conclusions seriously restrict the application of the operator factorization method in observability analysis and observer design for nonlinear control systems.

The present paper attempts to tackle the observer design problem for nonlinear control systems, by taking the system initial conditions into consideration, from the operator factorization approach. For this purpose, a new concept of dynamic right coprime factorization for nonlinear operators is first introduced, which is defined with respect to initial conditions. Basic properties of the dynamic right coprime factorization are then discussed. The dynamic right coprime factorization is finally applied to the problem of observer design for nonlinear control systems.

The rest of the paper is organized as follows. The next section presents the definition of the dynamic right coprime factorization for nonlinear operators, and discusses its basic properties. Section 3 analyzes linear control systems using the new concept. As a motivation for the new observer design method is to be proposed and studied in Section 4. It will be shown that a nonlinear control system possessing a dynamic right coprime factorization is always observable in the classical sense. The system stabilization problem via an observer approach is then investigated in Section 5, and a detailed design example is studied in Section 6. Finally, Section 7 concludes the paper with some remarks and a further discussion.

## II. DYNAMIC RIGHT COPRIME FACTORIZATION

### A. The conventional definition of right coprime factorization

To start with, we recall the conventional definition of right coprime factorization of nonlinear operators (see (de Figueiredo & Chen, 1993) for more details).

Let $P$: $U \rightarrow Y$ be a nonlinear (i.e., not necessarily linear) operator, which is assumed to be causal from linear space $U$ to linear space $Y$. If there is a linear space $W$ and if there are two causal and stable operators $N$: $W \rightarrow Y$ and $D$: $W \rightarrow U$, with $D$ being invertible, such that $P = ND^{-1}$, then the pair of operators ($N$, $D$) is said to be a *right factorization* of $P$. Throughout this paper, we assume that $U$, $Y$ and $W$ contain proper normed linear subspaces $U_S$, $Y_S$ and $W_S$, called stable subspaces of $U$, $Y$ and $W$, respectively. The norms equipped in $U_S$, $Y_S$ and $W_S$ are all $\boldsymbol{L}^2$-norm, for example,

$$\| u \|_2 = \max_{1 \le i \le m} (\int_0^\infty | u_i(t) |^2 \mathrm{d}t)^{1/2} .$$

$U_S$ is then $\{\|u\|_2 < \infty\}$.

In general, we may let $U$, $Y$ and $W$ be finite- or infinite-dimensional subspaces of the so-called extended linear space $L_e^2[0,\infty)$ (see (de Figueiredo & Chen, 1993) for definition).

As is well known, for the vector $u(t)=[u_1(t)\ u_2(t)\ ...u_m(t)]^T$, for every $t \in R$ its Euclidean norm is

$$| u(t) |_{R^m} = \sqrt{\sum_{i=1}^m u_i^2(t)} .$$

Moreover, if there exist two causal and stable operators $S$: $Y \rightarrow W$ and $R$: $U \rightarrow W$, with $R$ being invertible, such that

$$SN + RD = \overline{M} \tag{2.1}$$

for a unimodular operator $\overline{M}$ on $W$, then ($N$, $D$) is said to be a *right coprime factorization* of $P$. Here, as usual, an operator is said to be unimodular if it is stable, invertible, and its inverse is also stable. Equation (2.1) is usually called the *Bezout* identity. Without loss of generality, we can let $\overline{M}=I$, the identity operator. If $N$ and $D$ is not coprime, then there exists a nonunimodular largest right cofactor, $H$, such that $D = D_1 H$ and $N = N_1 H$ (the theory of general factorization refers to (Jacobson 1973)), where the operator $D_1$ is invertible and the pair ($N_1, D_1$) is coprime. There were several methods to obtain the largest cofactor $H$ (to see, for example, (Chen & Figueiredo 1992; Desoer & Kabuli, 1988).

If ($N$, $D$) is a right coprime factorization of $P$, and $S$ and $R$ are the operators that satisfy the *Bezout* identity, then we can construct a system as shown in Fig. 1.

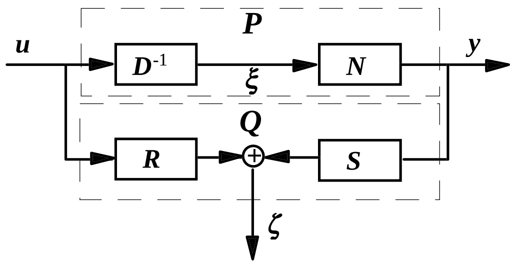


**Fig. 1** A compensated system

In Fig. 1, $Q$ stands for the compensator composing of $R$ and $S$, and $\zeta$ is its output. It follows easily, from Fig. 1 and Eq. (2.1), the following identity:

$$\zeta = Ru + Sy = RD\xi + SN\xi = \overline{M}\xi. \tag{2.2}$$

Thus, if $\overline{M}=I$, then Eq. (2.2) leads to

$$\zeta = \xi. \tag{2.3}$$

### B. A new definition of right coprimeness for dynamic systems

We now consider the case where the operator $P$ represents a *dynamic* plant whose input space is $U$ and output space is $Y$. It is known, from the state-space method, that there is a linear space $W$ whose elements are states of the plant $P$ such that the output, $y(t) \in Y$, depends not only on the input $u(t) \in U$ but also on the initial value of the state variable (or vector), $\xi(t) \in W$, and its differentials. Let $W_0$ be the finite-dimensional linear space that consists of all the initial states of the plant. The number of the dimension of $W_0$ may be larger than $W$. Let $\xi_0$ denote the initial condition that leads to a unique solution of $P$. Then $P$ is more precisely a mapping from $W_0 \times U$ to $Y$, i.e.,

$$y(t) = P(\xi_0, u(t)). \tag{2.4}$$

This can be understood from that a general state-space model of $P$ is

$$\begin{cases}\dot{\xi}(t) = f(\xi(t), u(t)) \\ \quad \xi(0) = \xi_0 \\ y(t) = h(\xi(t), u(t)),\end{cases} \tag{2.5}$$

which gives $\xi(t)=\xi(t;\ \xi_0,\ u(t))$. Then, $y(t)=y(t;\xi_0,\ u(t))$. Namely, $y$ is a function of $\xi_0$ and $u(t)$ with the time variable $t$.

Note that space $W_0{\times}U$ contains a stable subspace $W_0{\times}U_s$ whose norm is defined by $\|[\xi_0 u]^T\| = \max(|\xi_0|, \|u\|_2)$ for every $[\xi_0\ u]^T \in W_0{\times}U_s$.

Because the domain of the *dynamic* plant, $P$, involves the initial state subspace, the definition of the conventional right factorization of $P$ has to be modified appropriately.

**Definition 2.1** (Right factorization of a dynamic plant) Let $P$ be a dynamic plant. If there exist two stable dynamic operators $N$: $W{\times}S_N \to Y$ and $D$: $W{\times}S_D \to U$, where $S_N$ and $S_D$ are the initial state spaces of $N$ and $D$, respectively, and $D$ is an invertible operator with $D^{-1}$: $W_0{\times}U \to W$ such that for every $u(t)\in U$ and every $\xi_0 \in W_0$,

$$y(t) = P(\xi_0, u(t)) = ND^{-1}(\xi_0, u(t))$$
$$= N[D^{-1}(\xi_0, u(t))]. \tag{2.6}$$

Then the pair of operators ($N$, $D$) is said to be a *dynamic right factorization*, or simply, *D-right factorization*, of $P$. □

A plant with $D$-right factorization is illustrated in Fig. 2.

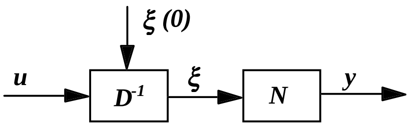


**Fig. 2** A plant described by a $D$-right factorization

The following remarks are given to Definition 2.1.

**Remark 2.1** As mentioned before, the dimension of $W_0$ is generally larger than that of $W$. This fact can be seen from the following linear equation $(b_0 \neq 0)$:

$$\xi''' + a_1\xi'' + a_2\xi' + a_3\xi = b_0u' + b_1u \tag{2.7}$$

To solving $\xi(t)$ from Eq. (2.7), it is necessary to provide $u(t)$ and the initial conditions

$$\xi(0) = a,\ \xi'(0) = b \text{ and } \xi''(0) = c.$$

The dimension of $W_0$ (or $\xi_0$) is 3, however, the dimension of $W$ is 1.

**Remark 2.2** Consider Eq. (2.7) again. If the problem is to solve $\xi(t)$ for a given $u(t)$, then Eq. (2.7) defines a mapping $D_1 : W_0 \times U \to W$; if $\xi(t)$ is known, the problem is to solve $u(t)$, then Eq. (2.7) defines another mapping $D_2 : W \times U_0 \to U$. Obviously, $D_1$ is the inverse of $D_2$, but the domain and codomain of $D_1$ hold different dimensions. It may be helpful to define $D_1 : W_0 \times U \to W \times U_0$. However, usually, we apply the former notation because the initial value $u_0$ is uniquely determined by $u(t)$.

**Remark 2.3** Let $y(t) = P(\xi_0, u(t)) = ND^{-1}$ ($\xi_0$, $u(t)$) be a D-right factorizaition of $P$. We then assume the mapping $N$ is relaxed, i.e., if $\xi(t)$=0, then $y(t)$=0. It implies that in the D-right factorizaition of $P$, all initial conditions of $N$ are zero. The hypothesis is reasonable because of causality.

In Definition 2.1, the operator $D^{-1}$ is a mapping from $W_0{\times}U$ to $W$, i.e., $\xi(t)= D^{-1}(\xi_0,\ u(t))$, or, $[\xi_0,\ u(t)]^T = D\xi(t)$. Two operators can be defined from $D$. The first one is a *projection operator* $\pi_0$:

$$\pi_0 : W \to W_0;\quad \xi(t) \mapsto \xi(0).$$

The second is

$$\tilde{D} : W \to U;\quad \xi(t) \mapsto u(t).$$

$\tilde{D}$ is called the *reduced operator* of $D$. The following lemmas provide some fundamental properties of $\tilde{D}$.

**Lemma 2.1** Let $\tilde{D}_1$ and $\tilde{D}_2$ be the reduced operators of $D_1$ and $D_2$, respectively. Then $D_1= D_2$ if and only if $\tilde{D}_1=\tilde{D}_2$.

**Proof:** Instead of a direct verification of the lemma, we prove an equivalent statement that $\tilde{D}_1\neq\tilde{D}_2$ if and only if $D_1\neq D_2$.

If $D_1\neq D_2$, then there exists a $\xi(t)\in W$ such that $D_1\xi(t) \neq D_2\xi(t)$, i.e.,

$$\begin{bmatrix}\xi(0)\\ \tilde{D}_1\xi(t)\end{bmatrix} = D_1\xi(t) \neq D_2\xi(t) = \begin{bmatrix}\xi(0)\\ \tilde{D}_2\xi(t)\end{bmatrix}.$$

Hence, $\tilde{D}_1\xi(t)\neq\tilde{D}_2\xi(t)$.

On the other hand, if $\tilde{D}_1\neq\tilde{D}_2$, there exists a $\xi(t)\in W$ such that $\tilde{D}_1\xi(t)\neq\tilde{D}_2\xi(t)$. Therefore, $\begin{bmatrix}\xi(0)\\ \tilde{D}_1\xi(t)\end{bmatrix} \neq \begin{bmatrix}\xi(0)\\ \tilde{D}_2\xi(t)\end{bmatrix}$, i.e., $D_1\xi(t)\neq D_2\xi(t)$. □

Note that although $D$ is an invertible operator, $\tilde{D}$ is generally not. Nevertheless, we can prove the following result.

Let $W(0)$ be the subspace of $W$ defined by $W(0) = \{\xi(t); \xi(t) \in W \text{ and } \xi(0) = 0\}$.

**Lemma 2.2** Let $\tilde{D}$ be the reduced operator of $D$. Then $\tilde{D}$ is an invertible operator from $W(0)$ to $U$. □

Lemma 2.2 is a direct result of Definition 2.1, hence, its proof is omitted.

Four remarks on Lemmas 2.1 and 2.2 are in order.

**Remark 2.4** Let $\alpha$ be a constant and $W(\alpha)$ be the set of vectors defined by $W(\alpha)=\{\xi(t);\xi(t)\in W \text{ and } \xi(0)=\alpha\}$ Then $\tilde{D}$ is invertible from $W(\alpha)$ to $U$. □

**Remark 2.5** $\tilde{D}$ is usually not an invertible operator from $W$ to $U$. It is onto but not one-to-one. However, it follows from Remark 2.4 that for every $u(t)\in U$, the set $\tilde{D}^{-1}(u(t))$ has property that if $\xi_1(t)$, $\xi_2(t)\in \tilde{D}^{-1}(u(t))$ with $\xi_1(0)=\xi_2(0)$, then $\xi_1(t)=\xi_2(t)$ for all $t\geq 0$. □

**Remark 2.6** For a causal plant, the set of states has the semi-group property, i.e., if we denote $\xi(t)=\xi(t;\ \xi(0),\ u(t))$, then $\xi(t+t_0)=\xi(t;\ \xi(t_0),\ u(t+t_0))$ for all $t_0\geq 0$ and $t>0$. By using the semi-group property and Remark 2.5, it can be seen that if $\xi_1(0)\neq\xi_2(0)$, then $\xi_1(t)\neq\xi_2(t)$ for all $t>0$. □

**Remark 2.7** If we fix the initial value $\xi(0)=0$, then the plant $P$ reduces to an operator $\tilde{P}:U\rightarrow Y$ defined by

$$\tilde{P}(u(t))=P(0,u(t)).$$

$\tilde{P}$ is a traditional operator, hence it has a conventional right factorization $\tilde{P}=\hat{N}\hat{D}^{-1}$. From Lemma 2.1, $\hat{D}$ is identical to $\tilde{D}$. It follows from Lemma 2.1 again that there is a unique operator $D$ such that $\tilde{D}$ is its reduced operator. □

When $P$ is a dynamic plant, and when $R$ and $S$ are dynamic systems in Eq. (2.2), their initial conditions have to be considered. In this case, Eq. (2.2) becomes

$$\begin{aligned}\zeta(t)&=R(z_1(0),u(t))+S(z_2(0),y(t))\\&=R(z_1(0),\tilde{D}\xi(t))+S(z_2(0),N\xi(t))\\&=\hat{M}(z_1(0),z_2(0),\xi(t)),\end{aligned}\tag{2.8}$$

where $\hat{M}$ is generally different from the $\overline{M}$ in Eq. (2.2). $\hat{M}$ is an operator from $Z_{10}\times Z_{20}\times W$ to $W$, where $Z_{10}$ and $Z_{20}$ are the initial state-spaces of $R$ and $S$, respectively.

**Definition 2.2** (Dynamic right coprimeness)

Let $(N, D)$ be a $D$-right factorization of the dynamic plant $P$. If there are two stable operators $S$: $Z_{20}\times Y\rightarrow W$ and $R$: $Z_{10}\times Y\rightarrow W$ such that for any initial conditions $z_1(0)$ and $z_2(0)$, the operator $\hat{M}(z_1(0),z_2(0),\xi(t))$ is a unimodular operator, then $(N, D)$ is said to be a *dynamic right coprime factorization* of $P$, or simply, a *D-right coprime factorization* of $P$. □

**Remark 2.8** If Eq. (2.8) holds, then $\zeta(0)=z_1(0)+z_2(0)$. Therefore, a more precise statement is that $\hat{M}(z_1(0),$ $z_2(0),\xi(t))$ is a unimodular operator from $W(\xi(0))$ to $W(z_1(0)+z_2(0))$. □

We note that in Definition 2.2, the coprimeness of $N$ and $D$ is defined via $N$ and $\tilde{D}$. Hence, we have the following result.

**Lemma 2.4** $(N, D)$ is a $D$-right coprime factorization of $P$ if and only if $(N, \tilde{D})$ is a conventional right coprime factorization of $\tilde{P}$ for any the initial conditions $z_1(0)$ and $z_2(0)$. □

## III. AN EXAMPLE

In order to gain more insights about the definition of the $D$-right coprimeness given in the previous section, we consider as an example the case of general linear time-invariant systems.

Let $P$ be a linear time-invariant system with the state-space description

$$P:\qquad\begin{cases}\dot{x}=Ax+Bu\\ x(0)=x_0\\ y=Cx,\end{cases}\tag{3.1}$$

where $(A, B, C)$ is assumed to be completely controllable and completely observable. Instead of $\xi$, we have used the conventional notation $x$ to denote the state vector here for the linear system (3.1). The *Laplace* transformation of the state vector $x(t)$ is given by

$$x(s)=(sI-A)^{-1}x(0)+(sI-A)^{-1}Bu(s),\tag{3.2}$$

where $u(s)$ is the *Laplace* transformation of $u(t)$. Equation (3.2) leads to a graphic illustration of system (3.1) as shown in Fig. 3, where $v=Bu$.

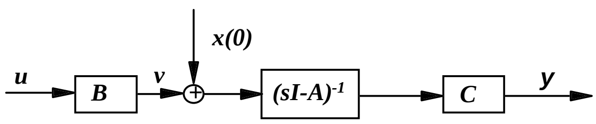


**Fig. 3** System (3.1).

Without loss of generality, we assume that the matrix $B$ has a full column rank (i.e., the column vectors of $B$ are linearly independent). Consequently, the space spanned by $v$ is isomorphic to $U$. Moreover, the observability of $(C, A)$ implies that $(C, (sI\text{-}A))$ is right coprime in the conventional sense.

System (3.1) is observable, so we can construct a *Luenberger* observer:

$$Q:\dot{\zeta}=(A-GC)\zeta+Bu+Gy,\tag{3.3}$$

where $G$ is a constant matrix such that $(A\text{-}GC)$ is *Hurwitz*. Solving Eq. (3.3) in the frequency domain, we obtain

$$\begin{aligned}\zeta(s)&=(sI-A+GC)^{-1}\zeta(0)\\&\quad+(sI-A+GC)^{-1}(v(s)+Gy(s)),\end{aligned}\tag{3.4}$$

where $v(s)$ and $y(s)$ are the *Laplace* transformations of $v(t)$ and $y(t)$, respectively. Figure 4 is obtained by combining Eqs. (3.2) and (3.4). It can be verified that the system shown in Fig. 4 is equivalent to that shown in Fig. 5, with $\zeta(0)=z_1(0)+z_2(0)$. Let

$$\overline{N}=C,\ \ \overline{D}=(sI\text{-}A),\ \ \overline{R}=(sI\text{-}A+GC)^{-1}\,\overline{N}$$

and

$$\overline{S}=(sI\text{-}A+GC)^{-1}G.\tag{3.5}$$

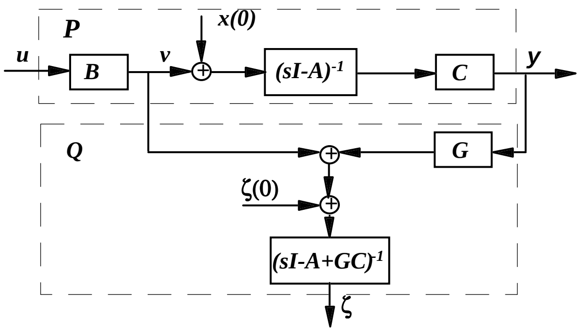


**Fig. 4** The linear plant with an observer.

Then, $\bar{R}$ and $\bar{S}$ are stable since *A-GC* is a *Hurwitz* matrix. Clearly, $\bar{R}$ is invertible. We can check that the *Bezout* identity holds. Indeed, we have

$$\bar{S}\bar{N}+\bar{R}\bar{D}=(sI-A+GC)^{-1}GC$$
$$+(sI-A+GC)^{-1}(sI-A)=I. \qquad (3.6)$$

Although the *Bezout* identity (3.6) holds for the linear system (3.1), Eq. (2.3) is generally not valid due to the initial conditions involved. This is the reason why we need the new definition of the *D*-right coprimeness for such dynamic plants.

Let us now take the initial conditions into account. From Fig. 5, we have

$$\begin{aligned}\zeta(s) &= (sI-A+GC)^{-1}(z_2(0)+Gy(s))\\ &\quad +(sI-A+GC)^{-1}(z_1(0)+v(s))\\ &= (sI-A+GC)^{-1}(z_2(0)+GCx(s))\\ &\quad +(sI-A+GC)^{-1}(z_1(0)+(sI-A)x(s)-x(0))\\ &= (sI-A+GC)^{-1}(z_1(0)+z_2(0)-x(0))+x(s).\end{aligned}$$

For any given $z_1(0)$, $z_2(0)$ and $x(0)$, $(sI\text{-}A+GC)^{-1}(z_1(0)+z_2(0)\text{-}x(0))$ is stable since (*A-GC*) is a *Hurwitz* matrix. Hence, $(sI\text{-}A+GC)^{-1}(z_1(0)+z_2(0)\text{-}x(0))+x(s)$ defines a unimodular operator from *W* to *W*. This can also be seen in the time domain, where we have

$$z(t)=x(t)+e^{(A-GC)t}[z_1(0)+z_2(0)-x(0)].$$

If the initial condition is considered, from the first equation of (3.1), the state vector should be

$$\begin{aligned}x(t) &= e^{At}x(0)+\int_0^t e^{A(t-\tau)}v(\tau)d\tau\\ &= [e^{At}\ \int_0^t e^{A(t-\tau)}(\bullet)d\tau]\begin{bmatrix}x(0)\\ v(t)\end{bmatrix},\end{aligned} \qquad (3.7)$$

where $\int_0^t a(t,\tau)(\bullet)d\tau$ is an operator defined by

$$[\int_0^t a(t,\tau)(\bullet)d\tau]v(t)=\int_0^t a(t,\tau)v(\tau)d\tau .$$

Therefore, we have

$$D^{-1}=[e^{At}\int_0^t e^{A(t-\tau)}(\bullet)d\tau] \text{ and } N=C.$$

It then follows from Eq. (3.7) that

$$\int_0^t e^{-A\tau}v(\tau)d\tau=e^{-At}x(t)-x(0).$$

Thus,

$$v(t)=e^{At}[\frac{d}{dt}e^{-At}x(t)].$$

Using similar notation as those used in Eq. (3.7), we have

$$D=\begin{bmatrix}\pi_0\\ e^{At}\ \dfrac{d}{dt}e^{-At}(\bullet)\end{bmatrix}.$$

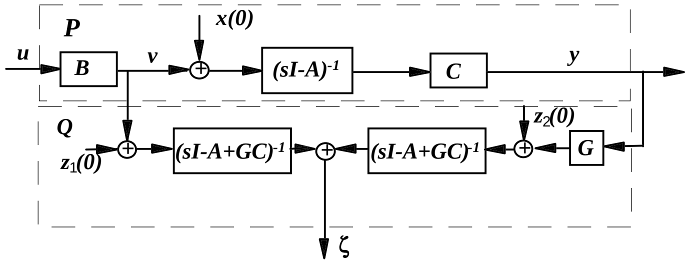


**Fig. 5** An equivalent configuration of Fig. 4.

Thus, $\tilde{D}=e^{At}\dfrac{d}{dt}e^{-At}(\bullet)$. Note that $\tilde{D}$ is an invertible operator from $W(\alpha)$ to $V$ ($V$=*BU*) for all real numbers α. By direct calculation, one can easily check that $D^{-1}D=I_1$ and $DD^{-1}=I_2$, where $I_1$ and $I_2$ are identity operators on spaces *W* and $W_0$X*V*, respectively.

It can be verified that $\tilde{D}(x(t))=\dfrac{d}{dt}x(t)-Ax(t)$ for all $x(t)\in W$. The formula for $\tilde{D}$ can also be obtained from the first equation of Eq. (3.1). If we deal with the *D*-right factorization for the linear system in the frequency domain, we actually have

$$D^{-1}=[(sI-A)^{-1}(sI-A)^{-1}],\ \ N=C$$

and $\tilde{D}=sI-A$.

The last two equations for *N* and $\tilde{D}$ are the same as those given in Eq. (3.4).

## IV. THE OBSERVER PROBLEM

This section is devoted to a formulation of the nonlinear observer problem from the *D*-right factorization approach and to a derivation of some conditions for the existence of an observer.

### A. Observability

For nonlinear systems, there are several definitions for observability. The definition adopted below was given by Hermann and Krener (1977), which is referred to the system described in Eq. (2.6).

**Definition 4.1** (Observability)

An initial condition $\xi(0)\in W_0$ is said to be *distinguishable* over *U* if for every $\bar{\xi}(0)\in W_0$ satisfying

$\bar{\xi}(0) \neq \xi(0)$, there is a $u(t)\in U$ such that $y(t;\xi(0), u(t)) \neq y(t;\bar{\xi}(0),u(t))$.

The plant $P$ is said to be *observable* if every $\xi(0)\in W_0$ is distinguishable over $U$. □

**Theorem 4.1** If the plant $P$ has a $D$-right coprime factorization, then $P$ is observable.

**Proof:** If $P$ has a $D$-right coprime factorization, then there are two operators $R$ and $S$ such that a system as shown in Fig. 1 can be constructed. Thus, it follows from Eq. (2.8) that

$$\begin{aligned}\zeta(t) &= R(z_1(0),u(t)) + S(z_2(0), y(t)) \\ &= \hat{M}(z_1(0), z_2(0), \xi(t)),\end{aligned} \tag{4.1}$$

where $\hat{M}$ is unimodular for any given initial conditions $z_1(0)$ and $z_2(0)$.

If the plant is not observable, on the other hand, then there exist two states $\xi_1(0)$ and $\xi_2(0)$, $\xi_1(0) \neq \xi_2(0)$, such that for all $u(t)\in U$ and for all $t\geq 0$, $y(t,\xi_1(0),u(t)) \equiv y(t,\xi_2(0),u(t))$. Thus, Eq. (4.1) implies that $\zeta(t;\xi_1,u(t)) = \zeta(t;\xi_2,u(t))$. Therefore, the operator $\hat{M}(z_1(0),z_2(0),\xi(t))$ is not one-to-one from $W$ to $W$ no matter what values of $z_1(0)$ and $z_2(0)$ are given. It contradicts the fact that $\hat{M}(z_1(0),z_2(0),\xi(t))$ is a unimodular operator. □

**B. Observers**

Consider a compensated system shown in Fig. 6 where $P$ is the plant with a right coprime factorization ($N$, $D$) and $Q$ is the compensator with inputs ($u$, $y$) and output $\zeta$. From Fig. 6, $y=N\xi(t)$ and $u=\tilde{D}\,\xi(t)$. Consequently, the inputs of $Q$ are functions of $\xi(t)$. Thus, when $Q$ is a dynamic compensator, the following equation is valid:

$$\zeta(t) = Q(t,\zeta(0),\xi(t)).$$

**Definition 4.2** (Observer)

The compensator $Q$ is said to be a *globally asymptotic observer*, or simply, an *observer* of the dynamic plant $P$ if for any $u(t)\in U$, the following conditions are satisfied:

1. when $\zeta(0)=\xi(0)$, $\zeta(t)= \xi(t)$ for all $t\in [0, \infty)$;
2. when $\zeta(0)\neq\xi(0)$, $|\zeta(t)-\xi(t)|_{\mathbf{R}^n} \to 0$ ($t\to\infty$).

$Q$ is said to be a *locally asymptotic observer*, or simply, a *local observer*, of the dynamic plant $P$ at $\xi(0)$ if there is a $\delta>0$, such that for any $u(t)\in U$, the following conditions are satisfied:

1. when $\zeta(0)=\xi(0)$, $\zeta(t)= \xi(t)$ for all $t\in [0, \infty)$;
2. when $\zeta(0)\neq\xi(0)$ and $|\zeta(0)-\xi(0)|_{\mathbf{R}^n}<\delta$, $|\zeta(t)-\xi(t)|_{\mathbf{R}^n} \to 0$ ($t\to\infty$). □

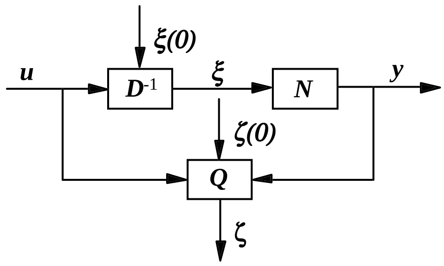


**Fig. 6** A nonlinear dynamic plant with a compensator.

**Definition 4.3** (Almost asymptotic observer)

The compensator $Q$ is said to be a *global almost asymptotic observer* (gaao) of the dynamic plant $P$ if for any $u(t)\in U$, the following conditions are satisfied:

1. when $\zeta(0)=\xi(0)$, $\zeta(t)= \xi(t)$ for all $t\in [0, \infty)$;
2. when $\zeta(0)\neq\xi(0)$, $|\zeta(t)-\xi(t)|_{\mathbf{R}^n} \to 0$ ($t\to\infty$) (a.a)*.

$Q$ is said to be a *local almost asymptotic observer* (laao) of the dynamic plant $P$ at $\xi(0)$ if there is a $\delta>0$, such that for any $u(t)\in U$, the following conditions are satisfied:

1. when $\zeta(0)=\xi(0)$, $\zeta(t)= \xi(t)$ for all $t\in [0, \infty)$;
2. when $\zeta(0)\neq\xi(0)$ and $|\zeta(0)-\xi(0)|_{\mathbf{R}^n}<\delta$, $|\zeta(t)-\xi(t)|_{\mathbf{R}^n} \to 0$ ($t\to\infty$) (a.a). □

It is obvious that a (global) observer is always an almost asymptotic observer. Conversely, if $P$ is a continuous-time system, i.e., $P$ is a continuous mapping and the input function $u(t)$ is also continuous, then gaao (laao) is a (local) observer.

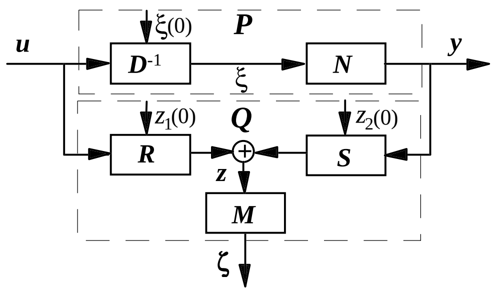


**Fig. 7** The configuration of a compensated system.

We now assume that the compensator $Q$ has a configuration as shown in Fig. 7, where $R$ and $S$ satisfy the *Bezout* identity and $M$ is a unimodular operator. Let $z_1(t)$ and $z_2(t)$ be the outputs of $R$ and $S$, respectively. It follows from Fig. 7 that $z(t) = z_1(t)+z_2(t)$; in particular, $z(0) = z_1(0)+z_2(0)$. Since the system is causal, we have $\zeta(0) =M(z(0))$.

**Lemma 4.1** Consider the system shown in Fig. 7. If the initial conditions of $R$ and $S$ can be adjusted, then there

* (a.a) means "almost all", i.e., the conclusion holds except on a subset of $[0,\infty)$ with zero Lebesque measure.

exists a unimodular operator $M_1$ such that $M = M_1^{-1}$ implies the output of $Q$, $\zeta(t)$ is equal to $\xi(t)$.

**Proof:** From Eq. (2.8), we have

$$\hat{M}(z_{10}, z_{20}, \xi(t)) = R(z_{10}, \tilde{D}\xi(t)) + S(z_{20}, N\xi(t)), \qquad (4.2)$$

where $\hat{M}(z_{10}, z_{20}, \xi(t))$ is unimodular for any initial values $z_{10}$ and $z_{20}$. Now, let us fix the values of $z_{10}$ and $z_{20}$, i.e., let $z_{10}=\bar{z}_1$ and $z_{20}=\bar{z}_2$. Then, we define an operator $M_1$ by

$$\begin{aligned} M_1:\quad & W(\xi_0) \to W(\alpha) \\ & \xi(t) \mapsto \hat{M}(\bar{z}_1, \bar{z}_2, \xi(t)), \end{aligned} \qquad (4.3)$$

where α is a constant depending only on $\xi(0)$, and $\bar{z}_1$ and $\bar{z}_1$ are constant vectors.

$M_1$ is a unimodular operator; hence, $M_1^{-1}$ exists and is stable. Let $M = M_1^{-1}$ in Fig. 7. Then the output of $Q$ satisfies

$$\zeta(t) = M_1^{-1}(z(t)) = M_1^{-1}[\hat{M}(z_{10}, z_{20}, \xi(t)]. \qquad (4.4)$$

When we designate $z_1(0)= z_{10}=\bar{z}_1$ and $z_2(0)= z_{20}=\bar{z}_2$, it follows that $\varsigma(t) = \xi(t)$. ▯

Lemma 4.1 gave an approach to reconstruct the state variable $\xi$ for a nonlinear system. However, it is difficult to use in practice. There are two problems. First, we generally cannot adjust the initial values of an observer. Consequently, we may fail to assign the initial values $\bar{z}_1$ and $\bar{z}_2$ to $R$ and $S$, respectively. Secondly, the operator $M_1^{-1}$ may contain the unknown initial value $\xi(0)$; hence, it varies when $\xi(0)$ changes. For example, for the linear case, we have obtained, in Section 3,

$$z(t) = x(t) + e^{(A-GC)t}[z_1(0) + z_2(0) - x(0)].$$

The operator $M_1$ is then defined by

$$M_1(x(t)) = x(t) + e^{(A-GC)t}[\bar{z}_1 + \bar{z}_2 - x(0)]. \qquad (4.5)$$

Consequently,

$$M_1^{-1}(z(t)) = x(t) - e^{(A-GC)t}[\bar{z}_1 + \bar{z}_2 - x(0)]. \qquad (4.6)$$

Note that the right-hand of Eq. (4.6) contains $x(0)$. Yet the goal of designing an observer is to estimate the state without any prior knowledge of $x(t)$. In the following discussion, we will drop the requirement on the initial value of the state and keep $M_1^{-1}$ invariant. In so doing, however, only a local result can be obtained.

We thereafter fix all the initial conditions: not only those of $z_{10}$ and $z_{20}$, but also $\xi(0)=\xi_0$. Then, by using the same notation as in Lemma 2.3 (note that the initial values have changed), we obtain a unimodular operator $M_2$ as follows:

$$M_2(\xi(t)) = \hat{M}(\bar{z}_1, \bar{z}_2, \xi(t)) = R(\bar{z}_1, \tilde{D}\xi(t)) + S(\bar{z}_2, N\xi(t)),$$

where $\xi(t) \in W(\alpha)$.

For continuous operators, we have the following results.

**Lemma 4.2** Consider the system shown in Fig. 7. Let $M = M_2^{-1}$. Then, there is a real number $\delta > 0$ such that when $|\xi(0)-\alpha|_{\mathbf{R}^n} < \delta$, there exists an initial value of $z(t)$, $z(0)=z(\xi(0))$, leading to $\zeta(t)=\xi(t)$ for all $t>0$.

**Proof**: From Eq. (4.2), we can define a function $H(z(0), \xi(t))$ as follows:

$$z(t) = H(z(0), \xi(t)) = \hat{M}(z_{10}, z_{20}, \xi(t)), \qquad (4.7)$$

where $z(0)=z_{10}+z_{20}$ is the initial value of $z(t)$. Then, define another function $\phi(z(0), \xi(t))$ by

$$\begin{aligned} \phi(z(0), \xi(t)) &= M_2^{-1}[H(z(0), \xi(t))] - \xi(t) \\ &= \zeta(t) - \xi(t). \end{aligned} \qquad (4.8)$$

If $\bar{z}(0) = \bar{z}_1 + \bar{z}_2$ and $\bar{\xi}(t) \in W(\alpha)$, then by repeating a process similar to the proof of Lemma 4.1, we obtain $\zeta(t) = \bar{\xi}(t)$, i.e., $\phi(\bar{z}(0), \bar{\xi}(t)) = 0$. Now, to apply the implicit function theorem, we only need to show that the function $\phi(z(0), \xi(t))$ is one-to-one with respect to the initial value $z(0)$. If $\hat{z}(0) \neq \tilde{z}(0)$, $H(\hat{z}(0), \xi(t)) \neq H(\tilde{z}(0), \xi(t))$, since $H(z(0), \xi(t))=z(t)$ whose initial value is $z(0)$. Hence, $M_2^{-1}H(\hat{z}(0), \xi(t)) \neq M_2^{-1}H(\tilde{z}(0), \xi(t))$, so that $\phi(\hat{z}(0), \xi(t)) \neq \phi(\tilde{z}(0), \xi(t))$.

By the implicit function theorem, we know that there are a real number $\delta > 0$ and a function $z(0)=z(\xi(t))$ (in fact, $z(0)=z(\xi(0))$ since the system is causal) such that if $|\xi(0) - \bar{\xi}(0)|_{R^n} < \delta$ then $\phi(z(0), \xi(t))=0$, namely, we have $\zeta(t)=\xi(t)$. ▯

**Theorem 4.2** Consider the system shown in Fig. 7. Let $M$ and $\delta > 0$ be defined as in Lemma 4.2. If $|\xi(0)-\alpha|_{\mathbf{R}^n} < \delta$ and $\xi(t) \in W_s$, then the compensator $Q$ in Fig. 7 is a local almost asymptotic observer.

Here, we should recall that $W_s$ is the stable subspace of $W$.

**Proof:** By Lemma 4.2, for every $\xi(t)$ which satisfies that $|\xi(0)-\alpha|<\delta$ and $\xi(t) \in W_s$, there is an initial value $\bar{z}(0) = z(\xi(0))$ such that $\zeta(t)=\xi(t)$. Moreover, since $M_2^{-1}H$ is a continuous operator for the initial condition $z(0)$, there exists a $\bar{\delta}$ such that when $|z(0) - \bar{z}(0)| < \bar{\delta}$, $M_2^{-1}H(z(0), \xi(t)) \in W_s$. Hence,

$$\zeta_i(t, z(0), \xi(t)) - \xi_i(t) \in W_s$$

for every $i \in \{1,2,..., n\}$, since the norm that we adopted for $W_s$ is the $L^2$-norm, we have

$$\int_0^\infty |\zeta_i(t, z(0), \xi(t)) - \xi_i(t)|^2 \, dt < \infty.$$

This implies that for any $\varepsilon$>0, there exists a $T$=$T$($\varepsilon$)>0 such that

$$\int_{T}^{\infty} |\zeta_i(t, z(0), \xi(t)) - \xi_i(t)|^2 \, \mathrm{d}t < \varepsilon.$$

Thus,

$$|\zeta_i(t, z(0), \xi(t)) - \xi_i(t)| \to 0 \ \ (t \to \infty) \ \ (a.a.)$$

As a result, the compensator $Q$ in Fig. 7 is a local almost asymptotic observer. □

**Corollary 4.1** Suppose that in the system (2.5), $y(t) = P(\xi_0, u(t))$ is continuous and has a $D$-right coprime factorization. Then this system has a local observer.

This corollary follows directly from Theorem 4.2 and the statement given after Definition 4.3.

**Corollary 4.2** Let $\delta$ be as defined in Lemma 4.2. Then, for any $\bar{\delta} < \delta$ and ε > 0, there exists a $T = T(\bar{\delta}, \xi(t), \varepsilon)$, independent of $z$(0), such that for all $\xi(t)$ satisfying $|\xi(0) - \alpha| \le \bar{\delta}$, we have

$$|\zeta(t, z(0), \xi(t)) - \xi(t)| < \varepsilon \quad (a.a.).$$

Because $|\xi(0) - \alpha| \le \bar{\delta}$ is a closed set, the conclusion follows from the Cantor Theorem in Calculus.

## V. STABILIZATION VIA OBSERVER

In the design of linear control systems, separability is an important performance index. With this property, a complicated problem can be decomposed into several simpler subproblems, so that the design is easily performed.

It is well known that the stabilization of a linear system implies the separability. Usually, the state of a control system is not accessible; hence, the stabilization design consists of constructing an observer and a feedback law. The separability asserts that these two problems in a design can be implemented individually. The composition of observer and feedback will assign the same poles as those designated for the feedback law. However, in general, the stabilization of a nonlinear system does not imply such a nice property. An example given by Pan *et al*., (1993) shows that a control system can be stabilized by a direct state feedback but fails if the feedback is given via an estimated state provided by an asymptotic observer.

However, it is interesting that the separability is still valid provided that the initial state errors can be controlled within a small area. This section will verify by using dynamic coprime factorization of nonlinear systems to show the separability. Consider a nonlinear plant $P$ described by a right coprime factorization ($N$, $D$). The state feedback used is $u$=$v$-$F$($\xi$) (as shown in Fig. 8).

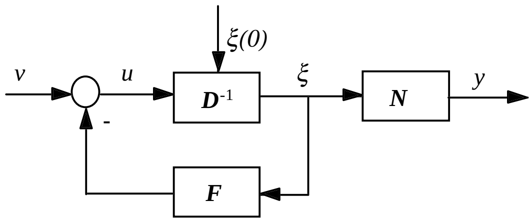


**Fig. 8** State feedback configuration.

**Theorem 5.1** Suppose that the closed loop system shown in Fig. 8 is well-posed (see Willems, 1971, for the definition of well-posedness) and the operator $F$ is stable. Then the closed-loop system is stable if and only if ($\tilde{D}$+$F$) is a unimodular operator, where $\tilde{D}$ is the reduced operator of $D$.

**Proof:** From Fig. 8, we have

$$u = \tilde{D}\xi = v - F\xi\,, \tag{5.1}$$

so that

$$v = (\tilde{D} + F)\xi\,. \tag{5.2}$$

Since the system is well-posed, $\xi$ is uniquely determined by $v$. The operator $\tilde{D}$+$F$ is then invertible. Consequently, for the closed-loop system, we have

$$y = N\xi = N(\tilde{D} + F)^{-1}v\,. \tag{5.3}$$

*Necessity.* Because the system is stable, $(\tilde{D} + F)^{-1}$ is stable. On the other hand, $\tilde{D}$ is stable since ($N$, $D$) is a right factorization. By assumption, $F$ is stable, so is $\tilde{D}$+$F$. Therefore, $\tilde{D}$+$F$ and $(\tilde{D} + F)^{-1}$ are both stable, namely, $\tilde{D}$+$F$ is unimodular.

*Sufficiency.* If $\tilde{D}$+$F$ is unimodular, then $(\tilde{D} + F)^{-1}$ is stable. It follows from Eq. (5.3) that the closed-loop system is stable. □

A system is said to be *overall stable* if when the input signal is stable (in the sense that it decays to zero asymptotically) then all the internal signals are also stable. In an overall stable closed-loop system, $F$ has to be stable. Thus, we have the following result, which is convenient to use.

**Corollary 5.1** Suppose that $P$ has a $D$-right coprime factorization ($N$, $D$). Then, a stable feedback $F$ can stabilize $P$ if and only if there exists a unimodular operator $M$ satisfying $F = M - \tilde{D}$.

**Proof:** By Theorem 5.1, the closed-loop system is stable if and only if $\tilde{D}$+$F$ is a unimodular operator. Denote $M$=$\tilde{D}$+$F$. Then the conclusion follows. □

Let us now consider the feedback configuration shown in Fig. 9, where $M_2 - \tilde{D} = F$ is the feedback determined in the stabilization design shown in Fig. 8, and operators $R$, $S$ and $M_1$ are those obtained in Section 4. From Section 4, we know that $Q$ is qualified as a local observer, i.e., for any given $\xi$(0) there exists $\bar{z}$(0)=$z$($\xi$(0)) such that $\zeta$($t$)=$\xi$($t$) and if $z$(0) is close enough to $\bar{z}$ (0), then ||ζ(*t*)-ξ(*t*)||→0. Note again that we have assumed that the operators given in Fig. 9 are all continuous.

**Theorem 5.2** Let $M_2 - \tilde{D}$ be the stabilizing feedback designed in Theorem 5.1, and let *Q* be the local observer at ξ(0). Then there exists a δ > 0 such that when $|z(0) - \bar{z}(0)| < \delta$ and $|\xi(0) - \bar{\xi}(0)| < \delta$, the system is stable.

**Proof:** Since the stable subspace $U_S$ is an open set, for every $v \in U_s$ there exists a $\delta_1$>0 such that for all $\Delta v$ with $|\Delta v| < \delta_1$, $v + \Delta v \in U_s$. $\tilde{D}$ is a continuous operator; hence, there is a $\delta_2$>0 such that when $|\Delta\xi| < \delta_2$, we have $||\tilde{D}(\xi + \Delta\xi) - \tilde{D}\xi|| < \delta_1$ .

Let *H*(*z*(0), ξ(*t*)) be as defined in Eq. (4.7). Observe that $H(\bar{z}(0), \xi(t)) = 0$ and $H(z(0), \xi(t))$ is continuous with respect to z(0). Hence, there exists a δ>0 such that $|\Delta\xi(t)| = |\zeta(t) - \xi(t)| < \delta_2$ provided $|z(0) - \bar{z}(0)| < \delta$. Thus, if $|z(0) - \bar{z}(0)| < \delta$, then we will have that $\zeta(t) = \xi(t) + \Delta\xi(t),\ \ \Delta\xi(t) \in U_s\ ,|| \Delta\xi(t) || < \delta$ .

The following equation can be obtained from an inspection of Fig. 9:

$$u = D\xi = v - (M_2 - \tilde{D})(\xi + \Delta\xi)\ .$$

It yields

$$M_2(\xi + \Delta\xi) = \tilde{D}(\xi + \Delta\xi) - \tilde{D}\xi + v\ . \qquad (5.4)$$

Since $||\Delta\xi|| < \delta_2$, $||\tilde{D}(\xi + \Delta\xi) - \tilde{D}\xi|| < \delta_1$ . Moreover, by the definition of $\delta_1$, the right-hand side of Eq. (5.4) is in the stable subspace, i.e., $\tilde{D}(\xi + \Delta\xi) - \tilde{D}\xi + v \in W_s$ . Since $M_2$ is a unimodular operator, $\xi + \Delta\xi \in W_s$. Note that we have required $\Delta\xi \in W_s$, which implies that $\xi \in W_s$. The conclusion of Theorem 5.2 then follows from that fact that all operators involved are stable. □

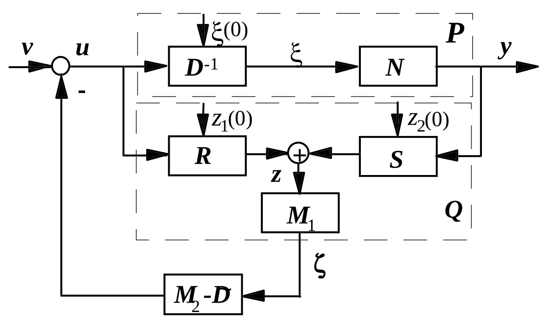


**Fig. 9** State feedback via an observer.

We note that in the proof of Theorem 5.2, we did not require $|\xi(t) - \zeta(t)| \to 0\ \ (t \to \infty)$. Hence, at the beginning of the proof we had no need to require $\xi \in W_s$.

## VI. AN EXAMPLE

As an illustrative example, we consider a nonlinear plant described by

$$y(t) = \xi(0) + \int_0^t u^{\frac{1}{3}}(\tau)d\tau\ , \qquad (6.1)$$

where ξ is the state of the plant and ξ(0) is the initial value of state.

The *D*-right factorization of the plant can be constructed as

$$\begin{aligned} &D : W \times S_D \to U \\ &u(t) = [-e^{-t}\xi_D(0) + \xi(t) - \int_0^t e^{-(t-\tau)}\xi(\tau)d\tau]^3 \end{aligned} \qquad (6.2)$$

and

$$\begin{aligned} &N: \quad WXS_N \to Y \\ &y(t) = e^{-t}\xi_N(0) + \int_0^t e^{-(t-\tau)}\xi(\tau)d\tau \end{aligned} \qquad (6.3)$$

where $\xi_D(0)$ and $\xi_N(0)$ are the initial values of *D* and *N*, respectively. Recall that we have assumed that $\xi_N(0) = 0$ . The inverse of *D* is

$$\begin{aligned} &D^{-1}: \quad UXW_0 \to W \\ &\xi(t) = \xi_{D^{-1}}(0) + u^{\frac{1}{3}}(t) + \int_0^t u^{\frac{1}{3}}(\tau)d\tau. \end{aligned} \qquad (6.4)$$

Obviously, $D^{-1}$ is an unstable operator. To check that $D^{-1}$ gives the exact inverse of *D*, we compute

$$DD^{-1} : UXW_0 \to UXW_0$$

$$\begin{bmatrix} \xi_D(0) \\ u_D(t) \end{bmatrix} = \begin{bmatrix} \xi_{D^{-1}}(0) \\ [-e^{-t}(\xi_D(0) - \xi_{D^{-1}}(0)) + u_{D^{-1}}^{\frac{1}{3}}(t)]^3 \end{bmatrix}. \qquad (6.5)$$

To distinguish the input of *D* from that of $D^{-1}$, we have used $u_D$-1(*t*) and $u_D(t)$ as substitutes for the input *u*(*t*) in Eqs. (6.2) and (6.5), respectively. Now, let $\xi_{D^{-1}}(0) = \xi_D(0)$. Then, by Eq. (6.5), we have $u_{D^{-1}}(t)$ $= u_D(t)$. From the same approach, we obtain

$$\begin{aligned} &D^{-1}D : W \to W \\ &\xi_{D^{-1}}(t) = (\xi_{D^{-1}}(0) - \xi_D(0)) + \xi_D(t). \end{aligned} \qquad (6.6)$$

Hence, $DD^{-1}$ and $D^{-1}D$ are both identity mappings. Moreover, we have

$$\begin{aligned} &ND^{-1}: \quad UXW_0 \to Y \\ &y(t) = e^{-t}\xi_{D^{-1}}(0) + \xi_{D^{-1}}(0) + \int_0^t u_{D^{-1}}^{\frac{1}{3}}(\tau)d\tau. \end{aligned} \qquad (6.7)$$

We next take $\xi_{D^{-1}}(0) = \xi_N(0) = 0$. Then it follows that Eq. (6.7) is identical to Eq. (6.1). As a result, (*N*, *D*) is a *D*-right factorization of *P*.

To verify the coprimeness, let

$$\begin{aligned} &S : YXZ_{10} \to W \\ &\xi(t) = e^{-2t}z_1(0) - y(t) + \int_0^t e^{-2(t-\tau)}y(\tau)d\tau, \end{aligned} \qquad (6.8)$$

and

$$R : UXZ_{20} \to W$$

$$\xi_4(t) = -\frac{1}{2}u^3(t) - \frac{1}{2}[e^{-2t}z_2(0) - \int_0^t e^{-2(t-\tau)}u^3(\tau)d\tau]. \tag{6.9}$$

Thus,

$$SN + RD: \quad W \to W$$

$$\bar{\xi}(t) = e^{-2t}(z_1(0) - \frac{1}{2}z_2(0) + \frac{1}{2}\xi_D(0)) - \frac{1}{2}\xi_D(t). \tag{6.10}$$

*SN+RD* is a unimodular operator for any initial conditions $z_1(0)$ and $z_2(0)$ since $e^{-2t}$ is a stable integration factor. Let *S* and *R* be as defined above and choose $M=-2$. Then a global asymptotic observer as shown in Fig. 7 can be constructed as follows. Let the feedback stabilizer be $(2I-\tilde{D})$. Then, the output of the closed - loop system, $y(t)$, is given by

$$y(t) = -\frac{1}{2}te^{-t}\xi_D(0) + (3e^{-t} - 3e^{-2t} - te^{-t})\xi_{D^{-1}}(0)$$
$$+ te^{-t}z_1(0) - \tfrac{1}{2}te^{-t}z_2(0) + \tfrac{1}{2}\int_0^t e^{-(t-\tau)}u(\tau)d\tau.$$

This output is obviously stable (converges to zero as $u(t) \to 0$). Moreover, we can check that all signals in the closed-loop system are stable.

## VII. CONCLUSIONS

The dynamic behavior of a control system depends on both the initial state and the control input. Particularly, for nonlinear systems, these two factors cannot be handled separately in general. This paper has formulated a dynamic right coprime factorization approach, which can simultaneously handle the initial state and the control input for the purposes of system observation and stabilization. Some fundamental properties of the dynamic right coprimeness have also been studied, and the relations of the dynamic right coprime factorization with observability and observer design have both been established. It has also been verified that the dynamic coprimeness implies the observability and the *Bezout* identity can be applied to design a dynamic observer. In addition, it has been shown that the linear separability of feedback and observer can be extended, at least locally, to nonlinear systems. A linear system example has been used to verify that the new results are consistent with the existing ones, and a nonlinear example has been constructed to visualize all the new concepts and the observer design procedure. It is believed that the new approach developed in this paper captures the very nature of nonlinear dynamic control systems, and hence will be useful for other analysis, design, simulation and implementation purposes. Future work along the same line includes effective construction methods of the dynamic observers, particularly global ones, for general nonlinear control systems.

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