ON THE EXISTENCE OF NORM-ESTIMATORS FOR SWITCHED SYSTEMS

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***Abstract*— In this paper, we prove the existence of norm-estimators for switched nonlinear systems. The proof is based on an existing converse Lyapunov theorem for IOSS nonlinear systems, and on the association of the switched system with a nonlinear system with inputs and disturbances that take values in a compact set.**

***Keywords*— Norm estimators, Converse Lyapunov theorems, IOSS, Nonlinear switched systems.**

I. INTRODUCTION

Recently, the study of switched systems has received a great deal of attention, being the rapidly developing area of intelligent control an important source of motivation for this study. Informally, a switched system is a family of continuous-time dynamical subsystems and a rule that determines the switching between them. The recent paper (Liberzon and Morse, 1999) is a very interesting survey on the subject, where an updated account of results and open problems may be found.

This paper concerns itself with the following question, for a switched system: *is it possible to estimate, based on external information provided by past input and output signals,* the magnitude *of the internal state* $x(t)$ *at time* t?

State estimation is central to control theory as it arises in signal processing applications (Kalman filters), in stabilization based on partial information (observers), etc. An open question is the derivation of useful necessary and sufficient conditions for the existence of observers, *i. e.*, dynamical systems which provide an estimate $\hat{x}(t)$ which converges to the state $x(t)$ of the system of interest, using the information provided by the sets of past inputs and outputs, $\{u(\tau)\,,\,\tau \le t\}$ and $\{y(\tau)\,,\,\tau \le t\}$ respectively. In order to stabilize a system to an equilibrium (that we will assume with no loss of generality to be the origin) of an Euclidean space, it may suffice to have a norm-estimate, that is to say, an upper bound $\hat{x}(t)$ on the *magnitude* (norm) $|x(t)|$ of the state $x(t)$. Indeed, it is often the case (Jiang and Praly, 1992; Praly and Wang, 1996) that norm-estimates suffice for control applications. To be more precise, in the context of switched systems, one wishes that $\hat{x}(t)$ becomes an upper bound of $|x(t)|$ as $t \to \infty$ uniformly with respect to the switching signal (see next section for precise definitions). We are thus interested in *uniform norm-estimators* which, when driven by the input-output data generated by a switched system, produce such an upper bound $\hat{x}(t)$ irrespectively of the switching signal.

One obvious necessary property for the possibility of norm-estimation is that the origin must be uniformly (with respect to the switching signal) globally asymptotically stable with respect to the "subsystem" consisting of those states for which the input $u \equiv 0$ produces the output $y \equiv 0$. In this case the switched system is *uniformly zero-detectable.* However this property is not sufficient, since one should ask that, irrespectively of the switching signal, when inputs and outputs are small, states should also be small, and if inputs and outputs converge to zero as $t \to \infty$, states do too.

On the other hand, the notion of input-output-to-state stability (IOSS), introduced by Sontag and Wang (1997) for a system

$$\left\{\begin{array}{rcl} \dot{x} &=& f(x,u) \\ y &=& h(x), \end{array}\right. \tag{1}$$

resulted in an useful paradigm in the study of nonlinear detectability. In that paper the authors describe relationships between the existence of full state observers and the IOSS property.

In a recent paper Krichman *et al.* (2001) proved that system (1) is IOSS if and only if it admits a norm estimator (in a sense that will be made precise in Section 4). This result suggests that, for our purposes, the right notion of detectability is the generalization of the IOSS property to switched systems.

Since in (Krichman *et al.*, 2001) it was also shown that this result is, in turn, a consequence of a necessary and sufficient characterization of the IOSS property in terms of a smooth dissipation function (an uniform IOSS Lyapunov function), the problem that naturally appears in our context may be stated as follows: given a switched system with outputs whose component subsystems are each IOSS, find necessary and sufficient conditions for the IOSS of the switched system, uniformly with respect to the switching signals.

It can be easily shown that the sufficient conditions for IOSS of system (1) established by Krichman *et al.* (2001) carry over with little changes to the switched system. In fact, the existence of a common IOSS Lyapunov function implies that the switched system is IOSS for arbitrary switching, and that a norm-estimator for this system exists.

The question of the validity of the converse, that arises naturally as a byproduct of the question of the existence of the norm-estimator, originates the problem of the existence of such Lyapunov function, *i.e.*, of the existence of a converse Lyapunov theorem.

In this paper we obtain a converse Lyapunov theorem for a certain class of switched nonlinear systems. This theorem is in the spirit of the converse ones for asymptotic stability, input-to-state stability (ISS) and integral input-to-state stability (iISS) of switched nonlinear systems obtained in (Mancilla Aguilar and García, 2000) and (Mancilla Aguilar and García, 2001) respectively, since we also suppose that the index set is not endowed *a priori* with any topology. Following the ideas there developed, we base our approach on the association of the switched nonlinear system with a perturbed control system whose disturbances take values in a compact set. Once this association is performed, we obtain a straightforward proof of the converse theorem above by extending the results about IOSS obtained in (Krichman *et al.*, 2001).

The outline of the paper is as follows. In section II we give the basic definitions including those of switched system, uniform input-output-to-state stability (UIOSS), uniform input-output-to-state-stability Lyapunov functions, etc. In section III we associate a perturbed control system with outputs whose disturbances take values in a compact set, with a class of switched nonlinear systems, and based on this association we prove the converse Lyapunov theorem. In section IV we introduce the notion of norm-estimator, and prove its existence for that class of switched nonlinear systems. Finally, in section V we present some conclusions.

II. SWITCHED SYSTEMS AND UIOSS

Here we introduce some notations and definitions that will be used in the sequel. We denote with $\mathbb{R}^n$ the usual n-dimensional Euclidean space, and with $|\cdot|$ its Euclidean norm. $B_r^n \subset \mathbb{R}^n$ stands for the closed ball of radius r around the origin in $\mathbb{R}^n$. The set of measurable locally essentially bounded functions $\mathbf{w} : [0, +\infty) \to \mathbb{R}^m$ is denoted by $L_{\infty,e}^m$; for each $t \geq 0$ and each $\mathbf{w} \in L_{\infty,e}^m$ we denote with $\mathbf{w}_t$ the truncation of $\mathbf{w}$ at t, *i.e.*, $\mathbf{w}_t(\tau) = \mathbf{w}(\tau)$ if $\tau \leq t$, and $\mathbf{w}_t(\tau) = 0$ if $\tau > t$, and $\|\mathbf{w}_t\| = \text{ess sup}\,\{|\mathbf{w}(\tau)|, 0 \leq \tau \leq t\}$.
Following standard terminology (Hahn, 1965), a continuous function $\alpha : [0, T) \to \mathbb{R}$ is *positive definite* if $\alpha(0) = 0$ and $\alpha(r) > 0$ whenever $r > 0$, and is of class $\mathcal{K}$ if in addition it is strictly increasing. A $\mathcal{K}_\infty$*-class* function α is a function of class $\mathcal{K}$, defined in $[0, +\infty)$ that verifies $\lim_{r\to+\infty} \alpha(r) = +\infty$. A continuous function $\beta : [0, +\infty) \times [0, +\infty) \to [0, +\infty)$ is a $\mathcal{KL}$*-class* function if for any fixed t, $\beta(\cdot, t)$ is of class $\mathcal{K}$, and for a fixed r, $\beta(r, \cdot)$ is decreasing and $\lim_{t\to+\infty} \beta(r, t) = 0$.

Let the family $\mathcal{P} = \{(f_\sigma(x, u), h(x)),\ f_\sigma : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n,\ \sigma \in \Gamma,\ h : \mathbb{R}^n \to \mathbb{R}^p\}$, where Γ is an index set and f_σ are locally Lispchitz, $f_\sigma(0, 0) = 0\ \forall\sigma \in \Gamma$ and the *fixed* output function h, is of class $\mathcal{C}^1$ and $h(0) = 0$.

We will assume in the sequel, with no loss of generality, that there is a one to one correspondence between the elements of Γ and the pairs of $\mathcal{P}$, *i.e.*, given $\sigma \neq \sigma'$ elements of Γ, then $(f_\sigma, h) \neq (f_{\sigma'}, h)$. Given the family $\mathcal{P}$, we consider the switched system with outputs

$$\left\{ \begin{array}{rcl} \dot{x}(t) &=& f_s(x(t), \mathbf{u}(t)) \\ y(t) &=& h(x(t)), \end{array} \right. \tag{2}$$

where $x \in \mathbb{R}^n$, $\mathbf{u} \in L_{\infty,e}^m$ and s is a *switching signal*, *i.e.*, s is a piecewise constant function $s : [0, +\infty) \to \Gamma$; we will denote by $\mathcal{S}$ the family of the switching signals of a given switched system. Associated with each $s \in \mathcal{S}$ there is a sequence of real numbers $0 = t_0 < t_1 < \cdots < t_k < \cdots$ and a sequence of indexes $\sigma_0, \sigma_1, \cdots, \sigma_k, \cdots$ such that $s(t) = \sigma_k$ for all $t_k \leq t < t_{k+1}$. We recall that a trajectory of (2) corresponding to $s \in \mathcal{S}$, $\mathbf{u} \in L_{\infty,e}^m$ and originating from $\xi \in \mathbb{R}^n$, is a locally absolutely continuous curve $\eta : [0, T) \to \mathbb{R}^n$, such that $\eta(0) = \xi$ and $\dot{\eta}(t) = f_{\sigma_k}(\eta(t), \mathbf{u}(t))$ *a.e.* $t \in [t_k, t_{k+1}) \cap [0, T)$. Then, since the members of $\mathcal{P}$ are locally Lipschitz, for each $s \in \mathcal{S}$, each initial condition $\xi \in \mathbb{R}^n$ and each control $\mathbf{u} \in L_{\infty,e}^m$ there exists a unique maximally defined trajectory corresponding to s, ξ and $\mathbf{u}$. We denote this curve and its maximal interval of definition by $x(t, \xi, \mathbf{u}, s)$ and $[0, T_{\xi,\mathbf{u},s}^x)$ respectively. We will also use the notation $y(t, \xi, \mathbf{u}, s) := h(x(t, \xi, \mathbf{u}, s))$, and, when clear from the context, we will write $y(t)$ instead of $y(t, \xi, \mathbf{u}, s)$.

Definition II.1 The system (2) is *uniformly (with respect to $s \in \mathcal{S}$) input-output-to-state stable*, (UIOSS) if there exist a $\mathcal{KL}$-function β and $\mathcal{K}$-functions γ_1 and γ_2 such that, for each input $\mathbf{u} \in L_{\infty,e}^m$ and each $\xi \in \mathbb{R}^n$, it holds that

$$|x(t, \xi, \mathbf{u}, s)| \leq \max\{\beta(|\xi|, t), \gamma_1(\|\mathbf{u}_t\|), \gamma_2(\|y_t\|)\} \tag{3}$$

for each $t \in [0, T_{\xi,\mathbf{u},s}^x)$ and for all $s \in \mathcal{S}$.

It follows from the definition above that a necessary, but not sufficient, condition for the UIOSS of the switched system (2) is each subsystem

$$\left\{ \begin{array}{rcl} \dot{x}(t) &=& f_\sigma(x(t), \mathbf{u}(t)) \\ y(t) &=& h(x(t)), \end{array} \right.$$

of the family $\mathcal{P}$ be IOSS.

In order to establish sufficient conditions for the UIOSS of (2), we introduce the following:

Definition II.2 A positive definite radially unbounded smooth ($\mathcal{C}^\infty$) function $V : I\!R^n \to I\!R_{\geq 0}$ is a *common IOSS-Lyapunov function for* $\mathcal{P}$ if there exists a $\mathcal{K}_\infty$-function α and two $\mathcal{K}$-functions χ and γ that verify:

$$\nabla V(\xi) f_\sigma(\xi, u) \leq -\alpha(|\xi|) + \chi(|u|) + \gamma(|h(\xi)|) \qquad (4)$$

for all $\xi \in I\!R^n$, $\sigma \in \Gamma$ and $u \in I\!R^m$.

As pointed out in the Introduction, it can be easily shown, with similar arguments to those used in (Krichman *et al.*, 2001), that the existence of a common IOSS-Lyapunov function V for $\mathcal{P}$ assures that the system (2) is UIOSS, *i.e.*,

Theorem II.1 Suppose that there exists a common IOSS Lyapunov function for $\mathcal{P}$. Then the system (2) is UIOSS.

III. A CONVERSE LYAPUNOV THEOREM

The question naturally arises whether the converse to Theorem II.1 holds.
In order to assure the existence of a common IOSS-Lyapunov function for $\mathcal{P}$, we consider the family $\mathcal{P}^\star$ of mappings associated with $\mathcal{P}$, $\mathcal{P}^\star = \{f_\sigma : (f_\sigma, h) \in \mathcal{P}\}$ and assume that this family verifies the following condition:

C The family $\mathcal{P}^\star$ is uniformly locally Lipschitz, *i.e.*, for each $N \in I\!N$ there exists $l_N \geq 0$ such that

$$|f_\sigma(x, u) - f_\sigma(x', u')| \leq l_N(|x - x'| + |u - u'|)$$

for all $(x, u), (x', u') \in B_N^n \times B_N^m$ and all $\sigma \in \Gamma$.

The main theorem of this paper may be stated as follows:

Theorem III.1 Suppose $\mathcal{P}^\star$ satisfies **C**. Then if system (2) is UIOSS there exists a common IOSS-Lyapunov function V for $\mathcal{P}$.

The following result will be useful in the proof of Theorem III.1, since it establishes the connection between switched systems and perturbed controlled nonlinear systems.

Theorem III.2 Suppose $\mathcal{P}^\star$ verifies **C**. Then there exist a compact metric space D, an injective function $\iota : \Gamma \to D$ and a continuous function $F : I\!R^n \times I\!R^m \times D \to I\!R^n$ such that:

1. $\iota(\Gamma)$ is dense in D.
2. $F(x, u, d)$ is locally Lipschitz on (x, u) uniformly on d, *i.e.*, for each compact subset K of $I\!R^n \times I\!R^m$ there is some constant c_K so that $|F(x, u, d) - F(x', u', d)| \leq c_K(|x - x'| + |u - u'|)$, for all $(x, u), (x', u') \in K$, and all $d \in D$.
3. $F(x, u, \iota(\sigma)) = f_\sigma(x, u)$ for all $x \in I\!R^n$, all $u \in I\!R^m$ and all $\sigma \in \Gamma$.

Proof: Since the family $\mathcal{P}^\star$ is equibounded due to the condition **C** and to the fact that $f_\sigma(0, 0) = 0$ for all $\sigma \in \Gamma$, the theorem can be proved following the same steps of the proof of Theorem 3.2 of Mancilla Aguilar and García, (2000). ∎

Remark III.1 It is clear that when Γ is a finite set the family $\mathcal{P}^\star$ satisfies **C**. In this case we can take $D = \Gamma$ endowed with the discrete metric, ι the identity map and $F(x, u, \sigma) = f_\sigma(x, u)$ for all $x \in I\!R^n$, all $u \in I\!R^m$ and all $\sigma \in \Gamma$.

Consider F, D and ι as in Theorem III.2. We denote with $\mathcal{D} = \{\mathbf{d} : [0, \infty) \to D, \mathbf{d} \text{ measurable}\}$ and with $\mathcal{D}_\Gamma \subset \mathcal{D}$ the subset of elements of $\mathcal{D}$ that are piecewise constant $\iota(\Gamma)$-valued functions, and associate to the switched system (2) the following perturbed control system with outputs:

$$\begin{cases} \dot{z}(t) &= F(z(t), \mathbf{u}(t), \mathbf{d}(t)), \\ y(t) &= h(z(t)) \end{cases} \qquad (5)$$

where the state z evolves in $I\!R^n$, the output y in $I\!R^p$, the control $\mathbf{u}$ belongs to $L^m_{\infty,e}$ and the disturbance $\mathbf{d}$ to $\mathcal{D}$.

Given $\xi \in I\!R^n$, $\mathbf{u} \in L^m_{\infty,e}$ and $\mathbf{d} \in \mathcal{D}$, if we denote with $z(t, \xi, \mathbf{u}, \mathbf{d})$ the maximally defined trajectory of (5) originating from ξ with control $\mathbf{u}$ and disturbance $\mathbf{d}$, and with $[0, T^z_{\xi,\mathbf{u},\mathbf{d}})$ its maximal interval of definition then, as consequence of Theorem III.2, the following hold:

- For each $s \in \mathcal{S}$, if $\mathbf{d}_s = \iota \circ s$, then $\mathbf{d}_s \in \mathcal{D}_\Gamma$, $T^z_{\xi,\mathbf{u},\mathbf{d}_s} = T^x_{\xi,\mathbf{u},s}$ and $x(\cdot, \xi, \mathbf{u}, s) = z(\cdot, \xi, \mathbf{u}, \mathbf{d}_s)$.
- For each $\mathbf{d} \in \mathcal{D}_\Gamma$, there exists a unique switching signal $s_\mathbf{d}$ such that $\iota \circ s_\mathbf{d} = \mathbf{d}$.

Remark III.2 We conclude from Theorem III.2 that a switched system with outputs whose associated family $\mathcal{P}^\star$ verifies **C** can be considered as a perturbed control system with outputs whose disturbances are piecewise constant functions that take values in a dense subset of a compact metric space D.

The following result will be used in the sequel:

Proposition III.1 Suppose that $\xi \in I\!R^n$, $\mathbf{u} \in L^m_{\infty,e}$ and $\mathbf{d} \in \mathcal{D}$ and let $\{\mathbf{d}_n : n \in I\!N\} \subset \mathcal{D}$ such that $\lim_{n\to\infty} \mathbf{d}_n(\tau) = \mathbf{d}(\tau)$ *a.e.* Then if $z(\cdot, \xi, \mathbf{u}, \mathbf{d})$ is defined on $[0, t]$, $z(\cdot, \xi, \mathbf{u}, \mathbf{d}_n)$ is also defined on $[0, t]$ for n large enough and, in addition, $\lim_{n\to\infty} z(\tau, \xi, \mathbf{u}, \mathbf{d}_n) = z(\tau, \xi, \mathbf{u}, \mathbf{d})$ for all $\tau \in [0, t]$.

Proof: It follows with arguments similar to those in the proof of Proposition 3.1 in (Mancilla Aguilar and García, 2000). ∎
In the following lemma we show that the perturbed control system (5) has the same stability properties as the switched system (2).

Lemma III.1 Suppose the family $\mathcal{P}^\star$ verifies **C**. If the switched system (2) is UIOSS, then the perturbed control system (5) is uniformly (with respect to $\mathbf{d} \in \mathcal{D}$) IOSS, *i.e.*, there exist a function β of class $\mathcal{KL}$ and functions γ_1 and γ_2 of class $\mathcal{K}$ such that, for each input $\mathbf{u} \in L^m_{\infty,e}$ and each initial state $\xi \in I\!R^n$, it holds that

$$|z(t,\xi,\mathbf{u},\mathbf{d})| \leq \max\{\beta(|\xi|,t), \gamma_1(\|\mathbf{u}_t\|), \gamma_2(\|y_t\|)\}, \quad (6)$$

for all $t \in [0, T^z_{\xi,\mathbf{u},\mathbf{d}})$ and all $\mathbf{d} \in \mathcal{D}$.

Proof: Suppose that $\xi \in I\!R^n$, $\mathbf{u} \in L^m_{\infty,e}$ and $\mathbf{d} \in \mathcal{D}$. Due to the density of $\iota(\Gamma)$ in D, there exists a sequence $\{s_n : n \in I\!N\} \subset \mathcal{S}$ such that $\mathbf{d}_n = \iota \circ s_n \to \mathbf{d}$ *a.e.* (see (Mancilla Aguilar and García, 2000) for details).
Let t such that $z(\tau,\xi,\mathbf{u},\mathbf{d})$ is defined for all $\tau \in [0,t]$. Then due to Proposition III.1, for all $\tau \in [0,t]$, $\lim_{n\to\infty} x(\tau,\xi,\mathbf{u},s_n) = \lim_{n\to\infty} z(\tau,\xi,\mathbf{u},\mathbf{d}_n) = z(\tau,\xi,\mathbf{u},\mathbf{d})$.
If (3) holds, then $|z(t,\xi,\mathbf{u},\mathbf{d})| = \lim_{n\to\infty} |x(t,\xi,\mathbf{u},s_n)| \leq \max\{\beta(|\xi|,t), \gamma_1(\|\mathbf{u}_t\|), \gamma_2(\|y_t\|)\}$.

Remark III.3 We recall that a positive definite radially unbounded $\mathcal{C}^1$ function $V : I\!R^n \to I\!R$ is an *UIOSS-Lyapunov function for system* (5) if there exist a function α of class $\mathcal{K}_\infty$ and two functions χ and γ of class $\mathcal{K}$ verifying:

$$\nabla V(\xi)F(\xi,u,d) \leq -\rho(|\xi|) + \gamma(|h(\xi)|) + \chi(|u|)$$

for all $\xi \in I\!R^n$, all $d \in D$ and all $u \in I\!R^m$.

It follows that if **C** holds for the family $\mathcal{P}^\star$, then V is an UIOSS-Lyapunov function for system (5) with ρ and χ as above if and only if it is a common IOSS-Lyapunov function for $\mathcal{P}$.

Due to this last remark, Theorem III.1 is a corollary of the following:

Theorem III.3 Suppose system (5) is UIOSS. Then there exist a smooth UIOSS-Lyapunov function for it.

Proof:
Since system (5) is UIOSS then, due to Theorem 1 in Krichman *et al.*, 2001, (which holds, with minor modifications of its proof if the disturbance set is a compact metric space instead of $[-1,1]^p$, $p \in I\!N$), there exists a smooth UIOSS-Lyapunov function for (5). ∎

IV. NORM-ESTIMATORS

In this section we prove that the existence of a common IOSS Lyapunov function for the family $\mathcal{P}$ implies the existence of a norm-estimator for the switched system (2) in the following sense.

Definition IV.1 A (uniform with respect to the switching signal) *state norm-estimator* for a switched system (2) (briefly, a norm-estimator for (2)) is a pair (Σ,ρ), where ρ is a function of class $\mathcal{K}_\infty$ and Σ is a system

$$\Sigma \;:\; \dot\varphi = g(\varphi,u,y) \quad (7)$$

evolving in $I\!R^l$ and driven by the controls and outputs of (2), such that the following conditions are satisfied:
- $g : I\!R^l \times I\!R^m \times I\!R^p \to I\!R^l$ is continuous and locally Lispchitz on φ uniformly on (u,y).
- There exist $\mathcal{K}$-functions $\hat\gamma_1$ and $\hat\gamma_2$ and a function $\hat\beta$ of class $\mathcal{KL}$ such that for any initial state $\zeta \in I\!R^l$, all inputs $\mathbf{u} \in L^m_{\infty,e}$ and $\mathbf{y} \in L^p_{\infty,e}$, and any t in the interval of definition of the solution $\varphi(\cdot,\zeta,\mathbf{u},\mathbf{y})$, the following holds:

$$|\varphi(t,\zeta,\mathbf{u},\mathbf{y})| \leq \hat\beta(|\zeta|,t) + \hat\gamma_1(\|\mathbf{u}_t\|) + \hat\gamma_2(\|\mathbf{y}_t\|), \quad (8)$$

that is, (7) is ISS with respect to u and y (considered as inputs).
- There is a function $\beta \in \mathcal{KL}$ such that, for any pair of initial states ξ and ζ of (2) and (7) respectively, any control $\mathbf{u} \in L^m_{\infty,e}$ and any switching signal $s \in \mathcal{S}$, it holds that

$$|x(t,\xi,\mathbf{u},s)| \leq \beta(|\xi|+|\zeta|,t) + \rho(|\varphi(t,\zeta,u,\mathbf{y}_{\xi,\mathbf{u},s})|) \quad (9)$$

for all $t \in [0, T^x_{\xi,\mathbf{u},s})$), where $\mathbf{y}_{\xi,\mathbf{u},s}$ denotes the output trajectory of (2), that is, $y(t,\xi,\mathbf{u},s)$.

A. Construction of a norm-estimator for (2)

In order to construct a norm-estimator for (2), we need the following lemma whose proof is similar to the one of Lemma 5.2 of (Krichman *et al.*, 2001)

Lemma IV.1 Suppose V is a common IOSS-Lyapunov function for $\mathcal{P}$. Then there exists a $\mathcal{K}_\infty$-function θ and $\mathcal{K}$ - functions χ_1 and χ_2 such that the function $W = \theta \circ V : I\!R^n \to I\!R_{\geq 0}$ is of class $\mathcal{C}^1$, positive definite, radially unbounded and verifies

$$\nabla W(\xi).f_\sigma(\xi,u) \leq -W(\xi) + \chi_1(|u|) + \chi_2(|h(\xi)|) \quad (10)$$

for all $\xi \in I\!R^n$, $u \in I\!R^m$ and $\sigma \in \Gamma$.

The norm-observer is then obtained as follows.

Theorem IV.1 Suppose that $\mathcal{P}^\star$ verifies condition **C**. Then, if the switched system (2) is UIOSS, it admits a norm-estimator.

Proof: From Theorem III.1 and Lemma IV.1 there exists a positive definite and radially unbounded function W of class $\mathcal{C}^1$ that verifies (10). Due to the positive definiteness and radial unboundedness of W, there exist $\mathcal{K}_\infty$-functions α_1 and α_2 such that

$$\alpha_1(|\xi|) \leq W(\xi) \leq \alpha_2(|\xi|) \quad (11)$$

for all $\xi \in I\!R^n$. We assume, with no loss of generality that $r \leq \alpha_2(r)$ for all $r > 0$.
Consider the system Σ

$$\Sigma \;:\; \dot\varphi = -\varphi + \chi_1(|u|) + \chi_2(|y|), \quad (12)$$

with χ_1 and χ_2 as in (10).

Since system (12) is ISS with respect to u and y, (it can be seen as an exponentially stable linear system driven by the "input" $(\chi_1(|u|), \chi_2(|y|)))$, the inequality (8) trivially holds.

Let any initial states ξ and ζ of (2) and (12) respectively, any control $\mathbf{u} \in L^m_{\infty,e}$ and any switching signal $s \in \mathcal{S}$, and consider the resulting trajectory $(x(t), \varphi(t))$ of the composite system

$$\begin{cases} \dot{x}(t) &= f_s(x(t), \mathbf{u}(t)) \\ \dot{\varphi}(t) &= -\varphi(t) + \chi_1(|\mathbf{u}(t)|) + \chi_2(|h(x(t))|) \end{cases}$$

with initial condition $(x(0), \varphi(0)) = (\xi, \zeta)$. It is easy to see that the maximal interval of definition of this trajectory is $[0, T^x_{\xi,\mathbf{u},s})$; then, due to property (10), we have

$$\begin{aligned} \frac{d}{dt}(W(x(t)) - \varphi(t)) &= \nabla W(x(t)) f_s(x(t), \mathbf{u}(t)) \\ &\quad + \varphi(t) - \chi_1(|\mathbf{u}(t)|) - \chi_2(|h(x(t))|) \\ &\quad \leq -(W(x(t)) - \varphi(t)) \end{aligned}$$

$a.e.\, t \in [0, T^x_{\xi,\mathbf{u},s})$. Then,

$$\begin{aligned} &W(x(t)) - \varphi(t) \leq e^{-t}(W(\xi) - \zeta) \quad \Rightarrow \\ &\quad W(x(t)) \leq |\varphi(t)| + e^{-t}(\alpha_2(|\xi|) + |\zeta|) \leq |\varphi(t)| \\ &\quad\quad + e^{-t}(\alpha_2(|\xi|) + \alpha_2(|\zeta|)) \\ &\quad\quad \leq |\varphi(t)| + 2e^{-t}\alpha_2(|\xi| + |\zeta|), \end{aligned}$$

since $r \leq \alpha_2(r)$ and $\alpha_2(r) + \alpha_2(s) \leq 2\alpha_2(r+s)$ for all $r, s \geq 0$. Then, and due to (11),

$$\begin{aligned} |x(t)| &\leq \alpha_1^{-1}(W(x(t)) \\ &\leq \alpha_1^{-1}(|\varphi(t)| + 2e^{-t}\alpha_2(|\xi| + |\zeta|)) \\ &\leq \alpha_1^{-1}(2|\varphi(t)|) + \alpha_1^{-1}(4e^{-t}\alpha_2(|\xi| + |\zeta|)), \end{aligned}$$

as $\alpha_1^{-1}(r+s) \leq \alpha_1^{-1}(r) + \alpha_1^{-1}(s)$ for all $r, s \geq 0$. Finally, if we take $\rho(s) = \alpha_1^{-1}(2s)$ and $\beta(s,t) = \alpha_1^{-1}(4e^{-t}\alpha_2(s))$, ρ is of class $\mathcal{K}_\infty$, β is of class $\mathcal{KL}$ and

$$|x(t)| \leq \beta(|\xi| + |\zeta|, t) + \rho(|\varphi(t)|);$$

in consequence (9) holds and the proposed pair is a norm-estimator for (2). ∎

V. CONCLUSIONS

In this paper we have proved the existence of a norm-estimator for a certain class of IOSS switched systems as a byproduct of a converse Lyapunov theorem for this class of IOSS systems. The proof of the theorem is based on the association of these system with perturbed nonlinear control systems, and on simple generalizations of results on a converse Lyapunov theorem for IOSS systems with bounded time-varying perturbations.

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