EXACT TRAVELLING WAVE SOLUTIONS TO THE GENERALIZED KURAMOTO-SIVASHINSKY EQUATION

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Abstract—By using a special transformation, the new exact travelling wave solutions to the generalized Kuramoto-Sivashinsky equation are obtained.

Keywords—travelling wave solutions, solitary wave solutions, Kuramoto-Sivashinsky equation.

I. INTRODUCTION

In this paper, we consider the generalized Kuramoto-Sivashinsky equation (Yang, 1994):

\[ u_t + \beta u^a u_x + \gamma u^r u_{xx} + \delta u_{xxxx} = 0, \]

where \(a, \beta, \gamma, \delta, \tau \in R \) and \(a\beta\gamma\delta \neq 0\).

When \(a = \beta = 1\) and \(\tau = 0\), (1) reduces to the original Kuramoto-Sivashinsky (K-S) equation. The K-S equation was derived by Kuramoto (1978) for the study of phase turbulence in the Belousov-Zhabotinsky reaction. An extension of this equation to two or more spatial dimensions was then given by Sivashinsky (1977, 1980) in the study of the propagation of a frame front for the case of mild combustion. The K-S equation represents one class of pattern formation equation (Yang, 1994; Temam, 1988), and it also serves as a good model of bifurcation and chaos (Abdel-Gawad & Abdusalam, 2001; Li and Chen 2001, 2002).

As far as the travelling wave solutions are concerned, one can always use the transform

\[ u(x,t) = u(\xi), \quad \xi = x - ct, \]

where \(c\) is the wave velocity. The travelling wave solutions of (1) satisfy the following ordinary differential equation:

\[ -cu' + \beta uu' + \gamma r u'' + \delta u''' = 0, \]

In (Yang, 1994), using the ansatz (Bernoulli equation)

\[ u' = au + bv^n, \]

where \(a, b, n \in R, ab < 0\) and \(n \neq 1\), the exact travelling wave solution to (1) for \(\alpha = 3\tau = 9\) was obtained. In this presentation, we further introduce the following ansatz:

\[ u(\xi) = v^b(\xi), \quad v' = av + bv^n, \]

where \(abh \neq 0, n \neq 1\) and \(ab < 0\), and obtain a new exact solution for the equation.

From (5), one first gets

\[ u(\xi) = \left[-\frac{a}{2b}\tanh\left(\frac{n-1}{2}a(\xi - c_0)\right) - \frac{a}{2b}\right]^{1/b}, \]

in which \(c_0\) is an arbitrary constant. If \(h/(n-1) > 0\), (6) is the solitary wave solution connecting the two stationary states \(u = n\) and \(u = (-\frac{a}{b})^{1/(n-1)}\) (Lu et al., 1993). So, the relative orbit is a heteroclinic orbit.

Repeating some differential calculations, one can obtain the following formulas:

\[ v'' = (a + nbc)^{-1}v', \]

\[ v''' = a^2 + abn(n^2 + n + 1)v^{n-1} + 3ab^2n^2(2n-1)v^{2n-2} + b^2(n^2 - 1)(3n - 2)v^{3n-3} v'. \]

\[ u' = hv^{h-1}v', \]

\[ u'' = [h^2av^{h-1} + hh(n + 1)v^{n+h-2}]v', \]

\[ u''' = \{h^4a^2v^{h-1} + ha^2b(n + h - 1)[h^2(n + h - 1) + (n+2h-1)]v^{n+h-2} + 3hab^2(n + h - 1)^2(2n+h-2)v^{2n+h-3} + hh(n + 1)(2n + h - 2)(3n + h - 3)v^{3n+h-4}\}v'. \]

Then, by substituting the first formula of (5) and (9)-(11) into (3), one has

\[ \{(-ch + \delta h^4a^3)v^{h-1} + \beta h a v^{n+h+1} + \gamma h^2ax^{h+k+1} + \gamma h^2n^{h+k+1} + \gamma h^2(n + h - 1)v^{n+(r+1)h-2} + h^2b(n + h - 1)\}v^{3n+h-2} + hab^2(n + h - 1)^2 + (2n + h - 2)v^{2n+h-3} + hh(n + 1)(2n + h - 2)v^{3n+h-4}\}v'. \]
(3n + h - 3)v^{3n+h-4} \right) v' = 0 \,.

(12)

By furthermore comparing the same orders of \( v \), one can determine values of the parameters \( a, b, n \) and \( h \). However, to consider all possible cases is rather complicated. In order to keep the presentation short, only the following interesting cases with \( h = 1, n = 0 \) and \( n = 2 \) are considered here.

II. CASE \( h=1 \)

If \( h = 1 \), then \( u(\xi) = v(\xi) \) and \( u' = av + bn \), so that (12) is reduced to

\[
(\delta a^3 - c) + \beta u^a + \gamma a u + \gamma bu u + \gamma - n + 1 + 3\delta ab^2 n^2 (2n-1) a^{2n-2} + 3\delta b^3 (2n-1) \right) v^{3n-3} = 0. \tag{13}
\]

By comparing the same orders of \( u \), one finds the following situations.

(1) \( n = \frac{1}{2} \)

1a) \( \tau = 0 \), \( \alpha = -\frac{1}{2} \)

\[
\delta a^3 - c + \gamma a = 0, \quad \beta + \frac{2}{3} \gamma b + \frac{7}{8} \delta a^2 b = 0, \]

that is,

\[
b = -\frac{8\beta}{4\gamma + 7\delta a^2}, \quad c = \delta a^3 + \gamma a. \tag{14}
\]

In (14), \( a \) is a parameter. One should choose \( a \) such that \( ab < 0 \). The same should be done for the similar cases below.

1b) \( \tau = -\frac{1}{2} \), \( \alpha = -1 \)

\[
\delta a^3 - c = 0, \quad \beta + \frac{1}{\gamma} b = 0, \quad \gamma a + \frac{7}{8} \delta a^2 b = 0, \]

that is,

\[
a = \frac{4\gamma^2}{7\delta a^3}, \quad b = \frac{2\beta}{\gamma}, \quad c = \delta a^3. \tag{15}
\]

(2) \( n = \frac{3}{2} \)

If \( n = \frac{3}{2} \), then (13) can be translated into

\[
(\delta a^3 - c) + \beta u^a + \gamma a u + \gamma bu u + \gamma - n + 1 + 3\delta ab^2 n^2 (2n-1) a^{2n-2} + 3\delta b^3 (2n-1) \right) v^{3n-3} = 0. \tag{13}
\]

The following results are immediate.

2a) \( \tau = 0 \), \( \alpha = -\frac{3}{2} \)

\[
\delta a^3 - c + \gamma a = 0, \quad \beta + \frac{2}{3} \gamma b + \frac{38}{27} \delta a^2 b = 0, \quad \beta + \frac{4}{9} \delta ab^2 = 0. \]

Therefore, one can easily find that

\[
a = \pm 3 \left( -\frac{\gamma}{199} \right)^{\frac{1}{2}} (\gamma \delta < 0), \quad b = \frac{3}{2} \left( -\frac{\beta}{\delta a^3} \right)^{\frac{1}{2}} (\delta a^3 < 0), \quad c = \delta a^3 + \gamma a. \tag{16}
\]

(2b) \( \tau = -\frac{1}{2} \), \( \alpha = -\frac{1}{3} \)

\[
\delta a^3 - c = 0, \quad \beta + \frac{2}{3} \gamma b + \frac{4}{9} \delta ab^2 = 0, \quad \gamma a + \frac{38}{27} \delta a^2 b = 0, \quad \beta + \frac{4}{9} \delta ab^2 = 0, \}

so,

\[
a = \frac{9\beta}{10\gamma}, \quad b = -\frac{5\gamma^2}{3\delta a^3}, \quad c = \delta a^3. \tag{17}
\]

(2c) \( \tau = -\frac{1}{3} \), \( \alpha = -\frac{2}{3} \)

\[
\delta a^3 - c = 0, \quad \gamma a + \frac{38}{27} \delta a^2 b = 0, \quad \beta + \frac{2}{3} \gamma b + \frac{4}{9} \delta ab^2 = 0, \}

so,

\[
a = \frac{9\gamma^2}{361\delta a^3}, \quad b = \frac{5\gamma^2}{20\gamma}, \quad c = \delta a^3. \tag{18}
\]

Besides \( n = \frac{1}{2} \) and \( n = \frac{3}{2} \), one has the following case.

(3) \( \tau = -1, \alpha = 3n - 3 \)

For this case,

\[
\delta a^3 - c = 0, \quad \gamma a + \delta a^2 b n^2 + n + 1 = 0, \quad \gamma a + \delta a^2 b n^2 = 0, \quad \beta + \delta a^2 n(2n-1)(3n-2) = 0.
\]

From the second and the third equations, it follows that \( n = 4 \). So, if and only if \( n = 4, \beta = 3\gamma = 9 \), there exist real number solutions for \( a, b, c \), as

\[
a = \gamma \left( -\frac{5}{49.26} \right)^{\frac{1}{2}}, \quad b = \left( -\frac{\beta}{2800} \right)^{\frac{1}{2}}, \quad c = \frac{5\gamma^3}{10584\delta a^3}. \tag{19}
\]

This result is the same as that obtained in Yang (1994).

III. CASE \( n=0 \)

For \( n = 0 \), (12) is reduced to

\[
(\delta a^3 - c) v^{h-1} + \beta a^h b + \gamma a b^{h-1} + \gamma b h - 1 + v^{h-2} + \delta a^2 b(2n-1) (3h^2 - 3h + 1) v^{h-2} + 3\delta a^2 b^2 (h-1)^{2}. \tag{20}
\]

\[
(h-2) v^{h-3} + \delta b^2 (h-1)(2h-3) v^{h-4} = 0. \]

After considering the coefficients of some orders of \( v \), one has the following cases, with \( h = 1, h = 2 \) and \( h = 3 \), respectively.

(1) \( h = 1 \)

There exists one and only one sub-case with \( \alpha = \tau \neq 0 \) for \( h = 1 \) (Note: \( \alpha \neq 0 \)), as follows.
(1a) $\alpha = \tau \neq 0$,
\[
a = -\frac{\beta}{\gamma}, \quad c = \delta a^3.
\]  
(21)

(2) $h = 2$

For $h = 2$, one also has a sub-case.

(2a) $\tau = 0$, $\alpha = -\frac{1}{2}$

For this sub-case, one has
\[
76a^3b + \gamma b + \delta = 0, \quad 8\delta a^3 - c + 2\gamma a = 0.
\]

Thus,
\[
b = -\frac{\beta}{76a^2 + \gamma}, \quad c = 8\delta a^3 + 2\gamma a.
\]  
(22)

(3) $h = 3$

For $h = 3$, (20) can be changed to
\[
(27\delta a^3 - c)\beta + \beta^3 a + 3\gamma a^3 + 2 \gamma b^{\beta + 1} + 38\delta a^2 b + 12\delta a b^2 = 0,
\]
and only three cases exist, as follows.

(3a) $\tau = -\frac{1}{3}$, $\alpha = -\frac{2}{3}$

Here,
\[
27\delta a^3 - c = 0, \quad 3\gamma a + 38\delta a^2 b = 0, \quad \beta + 2\gamma b + 12\delta a b^2 = 0.
\]

Therefore,
\[
a = \frac{30\gamma^2}{361\delta^3}, \quad b = -\frac{193}{20\gamma}, \quad c = 27\delta a^3.
\]  
(24)

(3b) $\tau = -\frac{1}{3}$, $\alpha = -\frac{1}{3}$

It is clear that
\[
27\delta a^3 - c = 0, \quad \beta + 3\gamma a + 38\delta a^2 b = 0, \quad 2\gamma b + 12\delta a b^2 = 0.
\]

Hence,
\[
a = \frac{3\beta}{10\gamma}, \quad b = -\frac{5\gamma^2}{96\beta}, \quad c = 27\delta a^3.
\]  
(25)

(3c) $\tau = 0$, $\alpha = -\frac{2}{3}$

By the same reasoning, one has
\[
27\delta a^3 - c + 3\gamma a = 0, \quad 2\gamma b + 38\delta a^2 b = 0, \quad \beta + 12\delta a b^2 = 0,
\]
so $a, b, c$ are as below:
\[
a = \pm \left(-\frac{\gamma}{19b}\right)^{\frac{1}{2}}, \quad \left(\frac{\beta}{\delta a} < 0\right), \quad b = \mp \left(-\frac{\beta}{12\delta a}\right)^{\frac{1}{2}},
\]
\[
\left(\frac{\beta}{\delta a} < 0\right), \quad c = 27\delta a^3 + 3\gamma a.
\]  
(26)

For $h \neq 1$, $h \neq 2$, and $h \neq 3$, a comparison between the corresponding terms in (20) gives only one case, as follows.

(4) $\alpha = 3\tau = -\frac{2}{3}$

It follows that
\[
\delta a^3 h^3 - c = 0, \quad \delta a^2 b(h - 1)(3h^2 - 3h + 1) + \gamma ah = 0,
\]
\[
3\delta ab^2(h - 1)^2(h - 2) + \gamma b(h - 1) = 0,
\]
\[
\beta + 3\gamma a + 38\delta a^2 b = 0, \quad 2\gamma b + 12\delta a b^2 = 0.
\]

The second and the third equations of the above system give $h = -\frac{1}{2}$, $\alpha = 3\tau = 9$, the above system has real number solutions for $a, b$ and $c$, as
\[
a = -\frac{\gamma}{14\delta} \left(\frac{35\delta}{\beta}\right)^{\frac{1}{2}}, \quad b = \frac{3}{2} \left(\frac{\beta}{35\delta}\right)^{\frac{1}{2}}, \quad c = -\frac{\delta a^3}{27}.
\]  
(27)

IV. CASE $n=2$

Substituting $n = 2$ into (12) gives
\[
(-c + \delta b^3 a^3)h - \delta a^2 b(h + 1)(3h^2 + 3h + 1)v^h + 3\delta a b^2 h(h + 1)h + 2\gamma b h^2 + 3\delta a^2 h + 2\gamma b + 12\delta a b^2 = 0.
\]

(28)

(1) $h = -1$

For $h = -1$, there exist only one sub-case.

(1a) $\tau = \alpha \neq 0$

\[
a = \frac{\beta}{\gamma}, \quad c = -\delta a^3.
\]  
(29)

(2) $h = -2$

For $h = -2$, there exist two sub-cases.

(2a) $\tau = 0$, $\alpha = -\frac{1}{2}$

By the same reason, one has
\[-c - 8\delta a^3 - 2\gamma a = 0, \quad -7\delta a^2 b - \gamma b + \beta = 0,
\]
so,
\[
b = \frac{\beta}{7\delta a^2 + \gamma}, \quad c = -2\gamma a - \delta a^3.
\]  
(30)

(2b) $\tau = -\frac{1}{3}$, $\alpha = -1$

Similarly, one has
\[-c - 8\delta a^3 = 0, \quad -7\delta a^2 b - 2\gamma a = 0, \quad -\gamma b + \beta = 0,
\]
so,
\[
a = \frac{2\gamma^2}{7\delta a^2 + \gamma}, \quad b = \frac{\beta}{\gamma}, \quad c = -\delta a^3.
\]  
(31)

(3) $h = -3$
For $h = -3$, there exist three sub-cases.

(3a) $\tau = 0$, $\alpha = -\frac{2}{3}$

For this sub-case,

$$c + 27\delta a^3 + 3\gamma a = 0, \quad 38\delta a^2 b + 2\gamma b = 0, \quad 12\delta a b^2 - \beta = 0.$$  

The solutions are

$$a = \pm \left( \frac{-\gamma}{19\delta} \right)^{\frac{1}{6}} \left( \frac{\gamma}{\delta} < 0 \right), \quad b = \mp \left( \frac{\beta}{12\delta a} \right)^{\frac{1}{6}} \left( \frac{\beta}{\delta a} < 0 \right),$$

$$c = -27\delta a^3 - 3\gamma a. \quad (32)$$

(3b) $\tau = \alpha = -\frac{1}{3}$

Similarly,

$$c + 27\delta a^3 = 0, \quad 38\delta a^2 b + 3\gamma a - \beta = 0, \quad 12\delta a b^2 + 2\gamma b = 0.$$  

The solutions are

$$a = -\frac{3\beta}{10\gamma}, \quad b = \frac{5\gamma}{9\beta}, \quad c = -27\delta a^3. \quad (33)$$

(3c) $\tau = -\frac{1}{3}, \alpha = -\frac{2}{3}$

The following system is determined by the same reasoning:

$$c + 27\delta a^3 = 0, \quad 38\delta a^2 b + 3\gamma a = 0, \quad 12\delta a b^2 + 2\gamma b - \beta = 0,$$

so,

$$a = -\frac{30\gamma^2}{3615\beta}, \quad b = \frac{19\beta}{20\gamma}, \quad c = -27\delta a^3. \quad (34)$$

Besides $h = -1, h = -2$ and $h = -3$, there exists only one case, with $h = \frac{1}{3}, \alpha = 3\tau.

(4) $h = \frac{1}{3}, \alpha = 3\tau.$

It follows that

$$-c + \delta h^3 a^3 = 0, \quad \delta h^2 b(h+1)(3h^2+3h+1) + \gamma ha = 0,$$

$$3\delta ab^2(h+1)^2(h+2) + \gamma b(h+1) = 0,$$

$$3\delta b^4(h+1)(h+2)(h+3) + \beta = 0.$$  

The second and third equations in the above system give $h = \frac{1}{3}$. So, if and only if $h = \frac{1}{3}$ and $\alpha = 3\tau = 9$, parameters $a, b$ and $c$ are given by

$$a = \frac{\gamma}{2} \left( \frac{5}{4938\beta} \right)^{\frac{1}{6}}, \quad b = -3 \left( \frac{\beta}{280\delta} \right)^{\frac{1}{6}}, \quad c = \frac{5\gamma^3}{10584\delta}. \quad (35)$$

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