SUFFICIENT CONDITIONS FOR HURWITZ STABILITY OF MATRICES

W. ZHANG†, S.Q. SHEN‡ and Z.Z. HAN†

†*Department of Automation, Shanghai Jiao Tong University, 200240, Shanghai, China*
avee@sjtu.edu.cn, zzhan@sjtu.edu.cn
‡*School of Applied Mathematics, University of Electronic Science and Technology of China, 610054, Chengdu, Sichuan, China*
shuqianshen@126.com

***Abstract*— New sufficient conditions for the Hurwitz stability of a complex matrix are established based on the concept of α-diagonally dominance. These criteria depend only on the entries of a given matrix. Numerical examples are given to illustrate the applications of these criteria.**

***Keywords*— Hurwitz stability, H-matrices, α-diagonally dominance**

I. INTRODUCTION

Hurwitz stability plays a fundamental role in control theory since a time-invariant linear system is stable if and only if its system matrix is a Hurwitz matrix (Chen, 1998). Thus, checking the Hurwitz stability is important for control systems. Many researchers have considered the problem and lots of useful criteria for the Hurwitz stability have been established in the last two decades (see, e.g., Wang *et al.*, 1994; Naimark and Zeheb, 1997; Huang, 1998; Duan and Patton, 1998; Franze *et al.*, 2006).

Being similar to the Lyapunov methods for the stability of differential equations, there are two kinds of criteria for the Hurwitz stability of matrices. One is indirect method, i.e., the stability is checked by the eigenvalues. The indirect methods include computing Jordan canonical form, calculating the invariant factors, etc. Generally, it is not easy to complete these computations due to the computational complexity. Another is the so-called direct method which deals with the stability based on the entries of a given matrix directly (see, e.g., Wang *et al.*, 1994; Naimark and Zeheb, 1997; Huang, 1998; Carotenuto *et al.*, 2004; Franze *et al.*, 2006). For example, Routh array (Chen, 1998), Hurwitz criterion (Duan and Patton, 1998) and Lyapunov functions method (Cai and Han, 2006) are well-known direct methods.

One of the important direct methods is based on Geršgorin Theorem (see, e.g., Chap. 6 of Horn and Johnson, 1985a; Varga, 2004). The advantage is that the method can directly point out the location of eigenvalues of a square matrix on the complex plane (see, e.g., Wang *et al.*, 1994; Naimark and Zeheb, 1997; Huang, 1998). In the last fifty years, Geršgorin-like criteria made many contributions in linear systems theory. A lot of designing techniques have been developed by using Geršgorin theorem (see, e.g., Wang *et al.*, 1994; Naimark and Zeheb, 1997; Carotenuto *et al.*, 2004; Franze *et al.*, 2006). Recently, Huang (1998) presented several Geršgorin-like criteria for Hurwitz matrices. In this paper we will follow Huang's work to investigate the Geršgorin-like criterion. Several new sufficient conditions for Hurwitz stability will be developed. We would like to emphasize that we directly deal with complex matrices while most of the existing literature are focused on real matrices.

The paper is organized as follows. The preliminaries including notations, concepts and some lemmas are presented in Section II. The main results are given in Section III. In Section IV, several numerical examples are given to illustrate the applications of the results. The conclusions are drawn in Section V.

II. PRELIMINARIES

This section presents preliminaries of the paper that include notations, concepts and lemmas.

Let $\mathbb{C}^{n\times n}$ denote the set of $n\times n$ complex matrices. $N := \{1, 2, \ldots, n\}$. Let I denote the identity matrix with appropriate dimensions. Let $A = [a_{ij}] \in \mathbb{C}^{n\times n}$. Then A is said to be Hurwitz stable if its eigenvalues are all in the left-half side of the complex plane. A matrix is called *Hurwitz matrix* if it is Hurwitz stable (see, for example, Chen, 1998). For $i \in N$, we define

$$R_i(A) := \sum_{j=1, j\neq i}^{n} |a_{ij}|, \quad C_i(A) := \sum_{j=1, j\neq i}^{n} |a_{ji}|,$$

which denote the *deleted absolute row and column sums* of A, respectively (see, p. 344 of Horn and Johnson, 1985a). Without loss of generality, throughout this paper we assume that $R_i(A) > 0$ and $C_i(A) > 0$ for all $i \in N$. In fact, if $R_i(A) = 0$ or $C_i(A) = 0$ for some $i \in N$, then the investigation of Hurwitz stability of A reduces to that of an $(n-1)\times(n-1)$ matrix, i.e.,

$$\begin{bmatrix} a_{1,1} & \cdots & a_{1,i-1} & a_{1,i+1} & \cdots & a_{1,n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{i-1,1} & \cdots & a_{i-1,i-1} & a_{i-1,i+1} & \cdots & a_{i-1,n} \\ a_{i+1,1} & \cdots & a_{i+1,i-1} & a_{i+1,i+1} & \cdots & a_{i+1,n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n,1} & \cdots & a_{n,i-1} & a_{n,i+1} & \cdots & a_{n,n} \end{bmatrix},$$

which is obtained by deleting the i-th row and i-th column from A.

Let $\alpha \in [0,1]$ be a constant. Then we define

$$N_2^\alpha := \left\{ i \in N \,\middle|\, |a_{ii}| > R_i(A)^\alpha C_i(A)^{1-\alpha} \right\},$$

and

$$N_1^\alpha := N \backslash N_2^\alpha.$$

Notice that we have assumed that $R_i(A) > 0$ and $C_i(A) > 0$, so $R_i(A)^\alpha$ and $C_i(A)^{1-\alpha}$ are well-defined for all $\alpha \in [0,1]$. In the following, we review several useful concepts and conclusions.

Definition 1 (Berman and Plemmons, 1994). Let $A = [a_{ij}] \in \mathbb{C}^{n\times n}$. If $|a_{ii}| > R_i(A)$ $(C_i(A))$ for all $i \in N$, then A is said to be row (column) *strictly diagonally dominant*. If there exists a positive diagonal matrix D such that AD (DA) is row (column) strictly diagonally dominant, then A is said to be *generalized strictly diagonally dominant* (GSDD).

A matrix is called *H-matrix* if it is GSDD (see, e.g., Zhang and Han, 2006; Huang *et al.*, 2006). Clearly, the diagonal entries of an H-matrix are nonzero. There are numbers of equivalent conditions for H-matrix (see, Berman and Plemmons, 1994).

Definition 2. (Berman and Plemmons, 1994) Let $A = [a_{ij}] \in \mathbb{C}^{n\times n}$. Then $\mathcal{M}(A) = [m_{ij}]$ is said to be the *comparison matrix* of A if $m_{ii} = |a_{ii}|$, and $m_{ij} = -|a_{ij}|$ for all $i \neq j$, $1 \le i,j \le n$.

Definition 3. (Berman and Plemmons, 1994) Let $A = [a_{ij}]$ be a real square matrix with $a_{ii} > 0$ and $a_{ij} \le 0$, $i \neq j$, $1 \le i,j \le n$. Then A is an *M-matrix* if $A + \varepsilon I$ is nonsingular for arbitrary $\varepsilon \ge 0$.

It is known that A is an H-matrix if and only if the comparison matrix of A is an M-matrix (see, e.g., Chapter 6 of Berman and Plemmons, 1994).

Definition 4. Let $A = [a_{ij}] \in \mathbb{C}^{n\times n}$. If there exists an $\alpha \in [0,1]$ such that $|a_{ii}| > R_i(A)^\alpha C_i(A)^{1-\alpha}$ for all $i \in N$, then A is said to be *strictly α-diagonally dominant* (α-SDD).

The following conclusion states that a strictly α-diagonally dominant matrix is also an H-matrix.

Lemma 1. If $A \in \alpha$-SDD, then A is an H-matrix.

Proof. Let $\mathcal{M}(A) = [m_{ij}]$ be the comparison matrix of A. Since $A \in \alpha$-SDD, by Ostrowskii Theorem (Corollary 6.4.11 of Horn and Johnson, 1985a), $\mathcal{M}(A) + \varepsilon I$ is nonsingular for arbitrary $\varepsilon \ge 0$. Therefore, $\mathcal{M}(A)$ is an M-matrix, and it follows that A is an H-matrix. □

Lemma 2 (Huang, 1998). Let $A = [a_{ij}] \in \mathbb{C}^{n\times n}$ be an H-matrix. If the diagonal entries a_{ii}, $i = 1, 2, \ldots, n$, are all real and among a_{ii} there are p positive numbers and $n-p$ negative numbers. Then A has p eigenvalues with positive real parts and $n-p$ eigenvalues with negative real parts.

Let $A \in \mathbb{C}^{n\times n}$. Then $B = \frac{1}{2}(A + A^H)$ is a Hermitian matrix, where A^H denotes the conjugate transpose of A. Let $\lambda_{min}(B)$ and $\lambda_{max}(B)$ be the minimum and the maximum eigenvalues of B, respectively. We now can state the following conclusion.

Lemma 3 (Horn and Johnson, 1985b). Let $A \in \mathbb{C}^{n\times n}$ and $\lambda(A)$ be an arbitrary eigenvalue of A. Let $B = \frac{1}{2}(A + A^H)$. Then

$$\lambda_{min}(B) \le \mathrm{Re}\lambda(\mathrm{A}) \le \lambda_{\max}(\mathrm{B}),$$

where $\mathrm{Re}\lambda(\mathrm{A})$ denotes the real part of $\lambda(A)$.

If $N_2^\alpha = N$, then A is an H-matrix by Lemma 1. Moreover, if A is an H-matrix, then $N_2^\alpha \neq \varnothing$ by applying Ostrowskii Theorem (see Corollary 6.4.11 of Horn and Johnson, 1985a). Hence, throughout this paper we assume that $N_1^\alpha \neq \varnothing$ and $N_2^\alpha \neq \varnothing$.

III. MAIN RESULTS

This section presents the main results of the paper.

Theorem 1. Let $A = [a_{ij}] \in \mathbb{C}^{n\times n}$ and $a_{ii} < 0$ for each $i \in N$. If there exist an $\alpha \in [0,1]$ and $0 < x_i, y_i \le 1$ $(i \in N)$, such that

$$|a_{ii}| > \frac{1}{x_i^\alpha y_i^{1-\alpha}} \left(\sum_{j=1, j\neq i}^{n} |a_{ij}| x_j \right)^\alpha \left(\sum_{j=1, j\neq i}^{n} |a_{ji}| y_j \right)^{1-\alpha} \tag{1}$$

for all $i \in N_1^\alpha$, and

$$|a_{ii}| \ge \frac{R_i(A)^\alpha C_i(A)^{1-\alpha}}{x_i y_i}, \quad i \in N_2^\alpha, \tag{2}$$

then A is a Hurwitz matrix.

Proof. Note that, without loss of generality, we have assumed that for all $i \in N$

$$R_i(A) > 0 \quad \text{and} \quad C_i(A) > 0. \tag{3}$$

When $i \in N_1^\alpha$, we denote

$$\delta_i = \frac{|a_{ii}| x_i^\alpha y_i^{1-\alpha} - \left(\sum\limits_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij}| x_j \right)^\alpha \left(\sum\limits_{\substack{j=1 \\ j\neq i}}^{n} |a_{ji}| y_j \right)^{1-\alpha}}{\left(\sum\limits_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij}| x_j \right)^\alpha \left(\sum\limits_{\substack{j=1 \\ j\neq i}}^{n} |a_{ji}| y_j \right)^{1-\alpha}}. \tag{4}$$

From (1) and (3), we have $0 < \delta_i < +\infty$. Let

$$\phi_i = \delta_i \frac{\sum\limits_{j=1, j\neq i}^{n} |a_{ij}| x_j}{\sum\limits_{j\in N_2^\alpha} |a_{ij}|}, \quad \omega_i = \delta_i \frac{\sum\limits_{j=1, j\neq i}^{n} |a_{ji}| y_j}{\sum\limits_{j\in N_2^\alpha} |a_{ji}|},$$

where if $\sum_{j\in N_2^\alpha}|a_{ij}|=0$ (resp. $\sum_{j\in N_2^\alpha}|a_{ji}|=0$), then we define $\phi_i=+\infty$ (resp. $\omega_i=+\infty$). It is easy to get $\phi_i>0,\ \omega_i>0$.

Let us define two sets

$$N_x^\alpha=\{i\in N_2^\alpha \mid x_i=1\},$$

$$N_y^\alpha=\{i\in N_2^\alpha \mid y_i=1\}.$$

Thus, for $i\in N_1^\alpha$, there exists a positive number ε such that

$$0<\varepsilon<\min\left\{\min_{i\in N_1^\alpha}\phi_i,\ \min_{i\in N_1^\alpha}\omega_i\ ,\ \min_{i\in N_2^\alpha\setminus N_x^\alpha}(1-x_i),\ \min_{i\in N_2^\alpha\setminus N_y^\alpha}(1-y_i)\right\}. \quad (5)$$

Construct two positive diagonal matrices

$$D=\mathrm{diag}(d_1,d_2,\ldots,d_n),\quad E=\mathrm{diag}(e_1,e_2,\ldots,e_n),$$

where

$$d_i=\begin{cases} x_i, & i\in N_1^\alpha\cup N_x^\alpha\\ x_i+\varepsilon, & i\in N_2^\alpha\setminus N_x^\alpha\end{cases}$$

and

$$e_i=\begin{cases} y_i, & i\in N_1^\alpha\cup N_y^\alpha\\ y_i+\varepsilon, & i\in N_2^\alpha\setminus N_y^\alpha\end{cases}.$$

Define $G=[g_{ij}]=EAD$. In the following we will show that for each $i\in N$,

$$|g_{ii}|>R_i(G)^\alpha C_i(G)^{1-\alpha}. \quad (6)$$

First, let us consider the case of $i\in N_1^\alpha$. For each $i\in N_1^\alpha$, we have from (4) that

$$|a_{ii}|x_i^\alpha y_i^{1-\alpha}=(1+\delta_i)\left(\sum_{j=1,j\neq i}^n|a_{ij}|x_j\right)^\alpha \times\left(\sum_{j=1,j\neq i}^n|a_{ji}|y_j\right)^{1-\alpha}. \quad (7)$$

We now prove that inequality (6) is valid for $i\in N_1^\alpha$ by the following four cases.

Case 1: $\sum_{j\in N_2^\alpha}|a_{ij}|=\sum_{j\in N_2^\alpha}|a_{ji}|=0$. From (1), we can obtain

$$\begin{aligned}|g_{ii}|=y_i|a_{ii}|x_i&>y_i^\alpha\left(\sum_{j\in N_1^\alpha,j\neq i}|a_{ij}|x_j\right)^\alpha\\&\quad\times\left(\sum_{j\in N_1^\alpha,j\neq i}|a_{ji}|y_j\right)^{1-\alpha}x_i^{1-\alpha}\\&=R_i(G)^\alpha C_i(G)^{1-\alpha}.\end{aligned}$$

Case 2: $\sum_{j\in N_2^\alpha}|a_{ij}|=0,\ \sum_{j\in N_2^\alpha}|a_{ji}|\neq 0$. It implies for all $j\in N_2^\alpha$, $|a_{ij}|=0$. From (5), we know that $\omega_i>\varepsilon$, that is,

$$\delta_i\sum_{j=1,j\neq i}^n|a_{ji}|y_j>\varepsilon\sum_{j\in N_2^\alpha}|a_{ji}|.$$

Equivalently,

$$(1+\delta_i)\sum_{j=1,j\neq i}^n|a_{ji}|y_j>\sum_{j=1,j\neq i}^n|a_{ji}|y_j+\varepsilon\sum_{j\in N_2^\alpha}|a_{ji}|. \quad (8)$$

Thus (7) and (8) result in

$$\begin{aligned}|g_{ii}|&=y_i|a_{ii}|x_i\\&=y_i^\alpha\left((1+\delta_i)\sum_{j=1,j\neq i}^n|a_{ij}|x_j\right)^\alpha\\&\quad\times\left((1+\delta_i)\sum_{j=1,j\neq i}^n|a_{ji}|y_j\right)^{1-\alpha}x_i^{1-\alpha}\\&>y_i^\alpha\left(\sum_{j\in N_1^\alpha,j\neq i}|a_{ij}|x_j\right)^\alpha\\&\quad\times\left(\sum_{j=1,j\neq i}^n|a_{ji}|y_j+\varepsilon\sum_{j\in N_2^\alpha}|a_{ji}|\right)^{1-\alpha}x_i^{1-\alpha}\\&\geq\left(\sum_{\substack{j\in N_1^\alpha\cup N_y^\alpha\\ j\neq i}}|a_{ji}|y_j+\sum_{j\in N_2^\alpha\setminus N_y^\alpha}|a_{ji}|(y_j+\varepsilon)\right)^{1-\alpha}\\&\quad\times\left(\sum_{j\in N_1^\alpha,j\neq i}|a_{ij}|x_j\right)^\alpha x_i^{1-\alpha}y_i^\alpha\\&=R_i(G)^\alpha C_i(G)^{1-\alpha}.\end{aligned}$$

Case 3: $\sum_{j\in N_2^\alpha}|a_{ij}|\neq 0,\ \sum_{j\in N_2^\alpha}|a_{ji}|=0$. Using a similar argument to that in Case 2, we can verify that inequality (6) is valid.

Case 4: $\sum_{j\in N_2^\alpha}|a_{ij}|\neq 0,\ \sum_{j\in N_2^\alpha}|a_{ji}|\neq 0$. It follows from (5) that $\phi_i>\varepsilon$, that is,

$$\delta_i\sum_{j=1,j\neq i}^n|a_{ij}|x_j>\varepsilon\sum_{j\in N_2^\alpha}|a_{ij}|.$$

Equivalently,

$$(1+\delta_i)\sum_{j=1,j\neq i}^n|a_{ij}|x_j>\sum_{j=1,j\neq i}^n|a_{ij}|x_j+\varepsilon\sum_{j\in N_2^\alpha}|a_{ij}|. \quad (9)$$

From the inequalities (7) and (9), it is true that

$$\begin{aligned}
|g_{ii}| &= y_i|a_{ii}|x_i \\
&= y_i^\alpha \left((1+\delta_i) \sum_{j=1, j\neq i}^{n} |a_{ij}|x_j \right)^\alpha \\
&\quad \times \left((1+\delta_i) \sum_{j=1, j\neq i}^{n} |a_{ji}|y_j \right)^{1-\alpha} x_i^{1-\alpha} \\
&> y_i^\alpha \left(\sum_{j=1, j\neq i}^{n} |a_{ij}|x_j + \varepsilon \sum_{j\in N_2^\alpha} |a_{ij}| \right)^\alpha \\
&\quad \times \left(\sum_{j=1, j\neq i}^{n} |a_{ji}|y_j + \varepsilon \sum_{j\in N_2^\alpha} |a_{ji}| \right)^{1-\alpha} x_i^{1-\alpha} \\
&\geq y_i^\alpha \left(\sum_{j\in N_1^\alpha, j\neq i} |a_{ij}|x_j + \sum_{j\in N_x^\alpha} |a_{ij}|x_j \right. \\
&\quad \left. + \sum_{j\in N_2^\alpha \setminus N_x^\alpha} |a_{ij}|(x_j+\varepsilon) \right)^\alpha \\
&\quad \times \left(\sum_{j\in N_1^\alpha, j\neq i} |a_{ji}|y_j + \sum_{j\in N_y^\alpha} |a_{ji}|y_j \right. \\
&\quad \left. + \sum_{j\in N_2^\alpha \setminus N_y^\alpha} |a_{ji}|(y_j+\varepsilon) \right)^{1-\alpha} x_i^{1-\alpha} \\
&= R_i(G)^\alpha C_i(G)^{1-\alpha}.
\end{aligned}$$

From the above discussions, we conclude that the inequality (6) is valid for each $i \in N_1^\alpha$.

We now turn to the case of $i \in N_2^\alpha$. For each $i \in N_2^\alpha$, from the choice of ε and the construction of D and E, we have

$$0 < d_i, e_i \leq 1, \quad i \in N.$$

Now we prove that (6) is valid for $i \in N_2^\alpha$ by the following four cases.

Case 1: $i \in N_x^\alpha \cap N_y^\alpha$. It is easy to get

$$|g_{ii}| = |a_{ii}| > R_i(A)^\alpha C_i(A)^{1-\alpha} \geq R_i(G)^\alpha C_i(G)^{1-\alpha}.$$

Case 2: $i \in N_x^\alpha, i \notin N_y^\alpha$. Based on (2), we have

$$\begin{aligned}
|g_{ii}| &= (y_i+\varepsilon)|a_{ii}| \geq \varepsilon|a_{ii}| + R_i(A)^\alpha C_i(A)^{1-\alpha} \\
&> R_i(A)^\alpha C_i(A)^{1-\alpha} \geq R_i(G)^\alpha C_i(G)^{1-\alpha}.
\end{aligned}$$

Case 3: $i \notin N_x^\alpha, i \in N_y^\alpha$. By a similar argument to that in Case 2, we can verify that inequality (6) is valid.

Case 4: $i \notin N_x^\alpha, i \notin N_y^\alpha$. It follows from (2) that

$$\begin{aligned}
|g_{ii}| &= (y_i+\varepsilon)|a_{ii}|(x_i+\varepsilon) \\
&\geq \varepsilon^2|a_{ii}| + R_i(A)^\alpha C_i(A)^{1-\alpha} \\
&> R_i(A)^\alpha C_i(A)^{1-\alpha} \geq R_i(G)^\alpha C_i(G)^{1-\alpha}.
\end{aligned}$$

Hence, $|g_{ii}| > R_i(G)^\alpha C_i(G)^{1-\alpha}$ for all $i \in N$. Thus, G is strictly α-diagonally dominant, and it follows from Lemma 1 that A is an H-matrix. Since $a_{ii} < 0$ for all $i \in N$, by Lemma 2, A is a Hurwitz matrix. □

Let $A = [a_{ij}] \in \mathbb{C}^{n\times n}$. Then, for $i \in N$ and $\alpha \in [0,1]$, we define two real numbers as follows:

$$Q_i^\alpha(A) = \frac{R_i(A)^\alpha C_i(A)^{1-\alpha}}{|a_{ii}|},$$

$$K_i^\alpha(A) = \frac{|a_{ii}| + R_i(A)^\alpha C_i(A)^{1-\alpha}}{2|a_{ii}|}.$$

Using the above notations, the following corollaries can be obtained directly from Theorem 1.

Corollary 1. Let $A = [a_{ij}] \in \mathbb{C}^{n\times n}$ and $a_{ii} < 0$ for each $i \in N$. If there exists an $\alpha \in [0,1]$ such that for all $i \in N_1^\alpha$,

$$|a_{ii}| > \left(\sum_{j\in N_1^\alpha, \neq i} |a_{ij}| + \sum_{j\in N_2^\alpha} Q_j^\alpha(A)|a_{ij}| \right)^\alpha C_i(A)^{1-\alpha},$$

then A is a Hurwitz matrix.

Remark 1. If $\alpha = 1$, then Corollary 1 is exactly the Theorem 1 of Huang (1998).

Corollary 2. Let $A = [a_{ij}] \in \mathbb{C}^{n\times n}$ and $a_{ii} < 0$ for each $i \in N$. If there exists an $\alpha \in [0,1]$ such that for all $i \in N_1^\alpha$,

$$|a_{ii}| > R_i(A)^\alpha \left(\sum_{j\in N_1^\alpha, \neq i} |a_{ji}| + \sum_{j\in N_2^\alpha} Q_j^\alpha(A)|a_{ji}| \right)^{1-\alpha},$$

then A is a Hurwitz matrix.

Corollary 3. Let $A = [a_{ij}] \in \mathbb{C}^{n\times n}$ and $a_{ii} < 0$ for each $i \in N$. If there exists an $\alpha \in [0,1]$ such that for all $i \in N_1^\alpha$,

$$\begin{aligned}
|a_{ii}| > Q_i^\alpha(A) &\left(\sum_{j\in N_1^\alpha, \neq i} \frac{|a_{ij}|}{Q_j^\alpha(A)} + \sum_{j\in N_2^\alpha} Q_j^\alpha(A)|a_{ij}| \right)^\alpha \\
&\times \left(\sum_{j\in N_1^\alpha, \neq i} \frac{|a_{ji}|}{Q_j^\alpha(A)} + \sum_{j\in N_2^\alpha} |a_{ji}| \right)^{1-\alpha},
\end{aligned}$$

then A is a Hurwitz matrix.

Corollary 4. Let $A = [a_{ij}] \in \mathbb{C}^{n\times n}$ and $a_{ii} < 0$ for each $i \in N$. If there exists an $\alpha \in [0,1]$ such that for all $i \in N_1^\alpha$,

$$\begin{aligned}
|a_{ii}| > Q_i^\alpha(A) &\left(\sum_{j\in N_1^\alpha, \neq i} \frac{|a_{ij}|}{Q_j^\alpha(A)} + \sum_{j\in N_2^\alpha} |a_{ij}| \right)^\alpha \\
&\times \left(\sum_{j\in N_1^\alpha, \neq i} \frac{|a_{ji}|}{Q_j^\alpha(A)} + \sum_{j\in N_2^\alpha} Q_j^\alpha(A)|a_{ji}| \right)^{1-\alpha},
\end{aligned}$$

then A is a Hurwitz matrix.

Corollary 5. Let $A = [a_{ij}] \in \mathbb{C}^{n\times n}$ and $a_{ii} < 0$ for each $i \in N$. If there exists an $\alpha \in [0,1]$ such that for all $i \in N_1^\alpha$,

$$|a_{ii}| > K_i^\alpha(A)\left(\sum_{j\in N_1^\alpha,\neq i}\frac{|a_{ji}|}{K_j^\alpha(A)} + \sum_{j\in N_2^\alpha}|a_{ji}|\right)^{1-\alpha} \times\left(\sum_{j\in N_1^\alpha,\neq i}\frac{|a_{ij}|}{K_j^\alpha(A)} + \sum_{j\in N_2^\alpha}Q_j^\alpha(A)|a_{ij}|\right)^{\alpha},$$

then A is a Hurwitz matrix.

Corollary 6. Let $A = [a_{ij}] \in \mathbb{C}^{n\times n}$ and $a_{ii} < 0$ for each $i \in N$. If there exists an $\alpha \in [0,1]$ such that for all $i \in N_1^\alpha$,

$$|a_{ii}| > K_i^\alpha(A)\left(\sum_{j\in N_1^\alpha,\neq i}\frac{|a_{ij}|}{K_j^\alpha(A)} + \sum_{j\in N_2^\alpha}|a_{ij}|\right)^{\alpha} \times\left(\sum_{j\in N_1^\alpha,\neq i}\frac{|a_{ji}|}{K_j^\alpha(A)} + \sum_{j\in N_2^\alpha}Q_j^\alpha(A)|a_{ji}|\right)^{1-\alpha},$$

then A is a Hurwitz matrix.

By applying Lemma 3, we can obtain the following Theorem 2.

Theorem 2. Let $A = [a_{ij}] \in \mathbb{C}^{n\times n}$ and $\mathrm{Re}(a_{ii}) < 0$ for $i \in N$. Define $B = \frac{1}{2}(A + A^H)$. If B satisfies the conditions of Theorem 1 or Corollaries 1-6, then A is a Hurwitz matrix.

IV. NUMERICAL EXAMPLES

This section presents numerical examples to illustrate the applications of the conclusions established in Section III.

Example 1. Consider matrix

$$A = \begin{bmatrix} -2 & 0 & -1 \\ 2i & -1.9 & -1 \\ 1 & -i & -1.9 \end{bmatrix},$$

Let $\alpha = 0.5$. Then it is straightforward to check that $N_1^{0.5} = \{3\}$, $N_2^{0.5} = \{1,2\}$. Since

$$|a_{33}| - \left(\frac{R_1(A)^{0.5}C_1(A)^{0.5}}{|a_{11}|}|a_{31}| + \frac{R_2(A)^{0.5}C_2(A)^{0.5}}{|a_{22}|}|a_{32}|\right)^{0.5} C_3(A)^{0.5} \approx 0.0145 > 0,$$

by Corollary 1, A is a Hurwitz matrix.

However, since $N_1^1 = \{2,3\}, N_2^1 = \{1\}$, and

$$|a_{22}| - \frac{R_1(A)}{|a_{11}|}|a_{21}| - |a_{23}| = -0.1 < 0,$$

it fails to meet the conditions of Theorem 1 of Huang (1998) in this example.

Example 2. Consider matrix

$$A = \begin{bmatrix} -3 & 1 & 0 & -1 \\ 1 & -4 & i & -1 \\ 1 & -2i & -1.7 & 0 \\ -2i & -1 & 0 & -3 \end{bmatrix}.$$

Let $\alpha = 0.5$. Then $N_1^{0.5} = \{3\}, N_2^{0.5} = \{1,2,4\}$. Since

$$|a_{33}| - R_3(A)^{0.5}\left(Q_1^{0.5}(A)|a_{13}| + Q_2^{0.5}(A)|a_{23}| + Q_4^{0.5}(A)|a_{43}|\right)^{0.5} \approx 0.0881 > 0,$$

by Corollary 2, A is a Hurwitz matrix. However, if we denote $N_0^1 = \{i \in N \mid |a_{ii}| = R_i(A)\}$, then by a direct calculation we can obtain $N_0^1 = \{4\}, N_1^1 \setminus N_0^1 = \{3\}, N_2^1 = \{1,2\}$, and

$$|a_{33}| - \frac{R_1(A)}{|a_{11}|}|a_{31}| - \frac{R_2(A)}{|a_{22}|}|a_{32}| - |a_{34}| \approx -0.4667 < 0.$$

A does not satisfy the conditions of Theorem 1 of Huang (1998) in this case.

Example 3. Consider matrix

$$A = \begin{bmatrix} -4+i & 2 & i \\ -4 & -4+2i & 2 \\ -3i & 2 & -3.7 \end{bmatrix}.$$

The real part of its diagonal entries are negative. Let $B = \frac{1}{2}(A + A^H)$. Then we have

$$B = \begin{bmatrix} -4 & -1 & 2i \\ -1 & -4 & 2 \\ -2i & 2 & -3.7 \end{bmatrix}.$$

Let $\alpha = 0.5$. Then $N_1^{0.5}(B) = \{3\}, N_2^{0.5}(B) = \{1,2\}$. It is straightforward to check

$$|b_{33}| - K_3(B)^{0.5}\left(|b_{13}| + |b_{23}|\right)^{0.5} \times \left(Q_1^{0.5}(B)|b_{31}| + Q_2^{0.5}(B)|b_{32}|\right)^{0.5} \approx 0.0955 > 0.$$

Note that the diagonal entries of B are all negative real numbers. Therefore, B satisfies the conditions of Corollary 5. It follows from Theorem 2 that A is a Hurwitz matrix.

However, since A is a complex matrix and its diagonal entries are not all real, the criteria in Wang *et al.* (1994), Huang (1998), Pastravanu and Voicu (2004) are not valid in this case.

V. CONCLUSIONS

Based on the concept of α-diagonally dominance, this paper presents new criteria for Hurwitz stability. More precisely, several Geršgorin-like sufficient conditions for Hurwitz stability of matrices are developed. These conditions are simple since they only depend on the entries of a given matrix. Moreover, we deal with complex rather than real matrix which is considered by most of the existing literature. Further research work may consider, for example, the iterative algorithm and the necessary conditions for Hurwitz matrices.

VI. ACKNOWLEDGMENTS

The authors would like to thank the anonymous reviewers and the subject editor Prof. Jorge A. Solsona for their helpful suggestions and valuable comments. This work was supported by the National Natural Science Foundation of China (No. 60674024).

REFERENCES

Berman, A. and R.J. Plemmons, *Nonnegative Matrices in the Mathematical Sciences*, SIAM, Philadelphia (1994).

Cai, X.S. and Z.Z. Han, "Universal construction of control Lyapunov functions of linear systems," *Latin American Appl. Research*, **36**, 15-22 (2006).

Carotenuto, L., G. Franze and P. Muraca, "Computational method to analyse the stability of interval matrices," *IEE Proc.-Conrol Theory Appl.,* **151**, 669-674 (2004).

Chen, C.T., *Linear system theory and design* (3rd edition), Oxford University Press (1998).

Duan, G.R. and R.J. Patton, "A note on Hurwitz stability of matrices," *Automatica*, **34**, 509-511 (1998).

Franze, G., L. Carotenuto and A. Balestrino, "New inclusion criterion for the stability of interval matrces," *IEE Proc.-Conrol Theory Appl.,* **153**, 478-482 (2006).

Horn, R.A. and C.R. Johnson, *Matrix Analysis*, Cambridge University Press, Cambridge (1985a).

Horn, R.A. and C.R. Johnson, *Topics in Matrix Analysis*, Cambridge University Press, Cambridge (1985b).

Huang, T.Z., "Stability criteria for matrices," *Automatica*, **34**, 637-639 (1998).

Huang, T.Z., W. Zhang and S.Q. Shen, "Regions containing eigenvalues of a matrices," *Electronic J. Linear Algebra*, **15**, 215-224 (2006).

Naimark, L. and E. Zeheb, "An extension of the Levy-Desplanque theorem and some stability conditions for matrices with uncertain entries," *IEEE Trans. Circuits Syst. I*, **44**, 167-170 (1997).

Pastravanu, O. and M. Voicu, "Necessary and sufficient conditions for componentwise stability of interval matrix systems," *IEEE Trans. Automat. Control*, **49**, 1016-1021 (2004).

Varga, R.S., *Geršgorin and His Circles*, Springer-Verlag, Berlin (2004).

Wang, K., A. Mickel and D. Liu, "Necessary and sufficient conditions for the Hurwitz and Schur stability of interval matrces," *IEEE Trans. Automat. Control*, **39**, 1251-1254 (1994).

Zhang, W. and Z.Z. Han, "Bounds for the spectral radius of block H-matrices," *Electronic J. Linear Algebra*, **15** 269-273 (2006).

Received: April 3, 2007.
Accepted: November 9, 2007.
Recommended by Subject Editor Jorge Solsona.