NEW STABILITY CRITERIA FOR DISCRETE-TIME SYSTEMS WITH INTERVAL TIME-VARYING DELAY AND POLYTOPIC UNCERTAINTY

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Abstract— This paper is considered with the robust stability problem for linear discrete-time systems with polytopic uncertainty and an interval time-varying delay in the state. On the basis of a novel Lyapunov-Krasovskii functional, new delay-range-dependent stability criteria are established by employing the free-weighting matrix approach and a Jensen-type sum inequality. It is shown that the newly proposed criteria can provide less conservative results than some existing ones. Numerical examples are given to illustrate the effectiveness of the proposed approach.

Keywords— Delay-range-dependent stability; Lyapunov-Krasovskii functional; Discrete-time systems; Time-varying delay; Linear matrix inequality (LMI)

I. INTRODUCTION

Time-delays are frequently encountered in many fields of engineering systems such as long transmission lines in pneumatic systems, nuclear reactors, rolling mills, communication networks and manufacturing processes (Gu et al., 2003; Hale and Lunel, 1993; Su and Zhang, 2009). In general, the existence of delays in system models may induce instability or poor performance of the closed-loop schemes. Therefore, the stability problem of time-delay systems has attracted much attention during the past decades. Numbers of stability criteria have been established for various types time-delay systems. These criteria can be classified into two types: delay-dependent and delay-independent stability conditions; the former includes the information on the size of the delay, while the latter does not (Xu and Lam, 2008). Usually, delay-dependent stability conditions are less conservative than the delay-independent ones especially in the case when the delay is small. Therefore, in recent years many researchers have devoted to investigating delay-dependent stability of time-delay systems (see e.g., Gu et al., 2003; Xu and Lam, 2008; and the references therein).

Surveying in the literature, various approaches have been proposed to derive the delay-dependent stability conditions (Xu and Lam, 2008). For instance, the discretized Lyapunov-Krasovskii functional approach (Gu et al., 2003) and the descriptor system approach (Fridman and Shaked, 2002) together with the bounding techniques (Park, 1999 and Moon et al., 2001). Recently, the free-weighting matrix method (He et al., 2004a, 2004b) has been extensively used in deriving the delay-dependent criteria, which is very helpful to reduce the conservatism in existing stability criteria (He et al., 2007; Peng and Tian, 2008; Li et al., 2008). In Jiang and Han (2008), new stability criteria for uncertain systems with interval time-varying delay are proposed by introducing new Lyapunov-Krasovskii functional and employing an integral inequality (Han, 2005). However, it is worth mentioning that most of the delay-dependent stability results in the existing literature are concerned with norm-bounded uncertain continuous-time systems, while little attention has been paid to discrete-time case with polytopic uncertainty (Liu et al., 2006).

Recently, the delay-dependent stability problem for discrete-time systems with interval time-varying delay has been studied in Gao et al. (2004), Fridman and Shaked (2005), Jiang et al. (2005), and Gao and Chen (2007). Some delay-dependent stability criteria are established by employing the free-weighting matrix approach (Gao and Chen, 2007) or the descriptor system approach (Fridman and Shanked, 2005). Very recently, Zhang et al. (2008) presented an improved stability criterion by considering the useful terms ignored in the Lyapunov-Krasovskii functional of the previous literature. However, there is still room for further investigation. For example, in Zhang et al. (2008), the term $A^TPA^T$ is involved in the stability criterion. Therefore, it is not easy to extend the proposed criterion to polytopic-type systems. Moreover,
the criterion in Liu et al. (2006) for polytopic systems is actually not a delay-dependent stability condition since it only depends on the size of the interval.

In this note, we consider the delay-dependent stability of discrete-time systems with polytopic uncertainty and an interval time-varying delay in the state. Firstly, a novel Lyapunov-Krasovskii functional, which makes use of the information of both the lower and upper bounds of the interval time-varying delay, is introduced. Then, based on this functional, a new delay-range-dependent stability criterion is established for the nominal system in terms of linear matrix inequalities (LMIs). The criterion is easily adapted for the nominal system in terms of linear matrix inequalities. The criterion in Liu et al. (2006) for polytopic systems, which have been widely investigated in recent literature (see also Zhu and Yang, 2008). This inequality is helpful to derive the stability criteria.

Lemma 1 (Jiang et al. 2005). For any constant matrix $W \in \mathbb{R}^{n \times n}$, $W = W^T > 0$, two integers $r_M$ and $r_m$ satisfying $r_M \geq r_m$, vector function $x : [r_m, r_M] \rightarrow \mathbb{R}^n$, the following inequality holds:

\[
\left( \sum_{i=r_m}^{r_M} x(i) \right)^T W \left( \sum_{i=r_m}^{r_M} x(i) \right) \leq (r_M - r_m + 1) \sum_{i=r_m}^{r_M} x^T(i) W x(i).
\]

III. MAIN RESULTS

In this section, we first consider the stability for the nominal system described by (2), i.e., $A$ and $A_d$ are both known matrices. By introducing a new Lyapunov-Krasovskii functional, which makes use of the range information of the time-varying delay, we can establish the following result.

Theorem 1. The discrete time-delay system (2) is asymptotically stable for any time delay $d(k)$ satisfying (3), if there exist matrices $P > 0$, $Q_1 > 0$, $Q_2 > 0$, $R_1 > 0$, $R_2 > 0$, $T_1$, $T_2$, $T_3$, $L_T = [L^T_1, L^T_2, L^T_3]$, $M_T = [M^T_1, M^T_2, M^T_3]$ and $N_T = [N^T_1, N^T_2, N^T_3]$ of appropriate dimensions such that the following LMI holds:

\[
\Theta := \begin{bmatrix}
\Omega & M - L & -N & -L & -M & -N \\
* & Q_2 - Q_1 & 0 & 0 & 0 & 0 \\
* & * & -Q_2 & 0 & 0 & 0 \\
* & * & * & -R_1 & 0 & 0 \\
* & * & * & * & -R_2 & 0 \\
* & * & * & * & * & -R_2
\end{bmatrix} < 0,
\]

W. ZHANG, Q. Y. XIE, X. S. CAI, Z. Z. HAN
where \( \rho = d_M - d_m \),
\[
\Omega = \begin{bmatrix}
\Omega_{11} & \Omega_{12} & \Omega_{13} \\
* & \Omega_{22} & \Omega_{23} \\
* & * & \Omega_{33}
\end{bmatrix},
\]
with
\[
\begin{align*}
\Omega_{11} &= T_1 A + A^T T_1^T - T_1 - T_1^T + L_1 + L_1^T + Q_1, \\
\Omega_{12} &= T_1 A_d - M_1 + N_1 + L_2^T - A^T T_2^T, \\
\Omega_{13} &= P - T_1 + L_3^T - T_3^T + A^T T_3^T, \\
\Omega_{22} &= -M_2 - M_2^T + N_2 + N_2^T + T_2 A_d + A_2^T T_2^T, \\
\Omega_{23} &= -T_2 - M_3^T + N_3^T + A_3^T T_3^T, \\
\Omega_{33} &= P - T_3 - T_3^T + d_m^2 R_1 + \rho^2 R_2.
\end{align*}
\]

**Proof.** Choose a Lyapunov-Krasovskii functional candidate for the system (2) as follows
\[
V(k) = V_1(k) + V_2(k) + V_3(k) + V_4(k),
\]
where
\[
\begin{align*}
V_1(k) &= x^T(k) P x(k), \\
V_2(k) &= \sum_{i=k-d_m}^{k-1} x^T(i) Q_1 x(i) + \sum_{i=k-d_M}^{k-1} x^T(i) Q_2 x(i), \\
V_3(k) &= d_m \sum_{j=-d_m}^{k-1} \sum_{i=k+j}^{i-k-1} \eta^T(i) R_1 \eta(i), \\
V_4(k) &= \rho \sum_{j=-d_M}^{k-1} \sum_{i=k+j}^{i-k-1} \eta^T(i) R_2 \eta(i),
\end{align*}
\]
and \( P > 0, Q_1 > 0, Q_2 > 0, R_1 > 0, R_2 > 0 \) are matrices to be determined.

Let us define for \( i = 1, \ldots, 4 \),
\[
\Delta V_i(k) = V_i(k+1) - V_i(k).
\]

Note that \( x(k+1) = \eta(k) + x(k) \). Then, along the solution of system (2), we have
\[
\begin{align*}
\Delta V_1(k) &= \eta^T(k) P \eta(k) + \eta^T(k) P x(k) + x^T(k) P \eta(k), \\
\Delta V_2(k) &= x^T(k) Q_1 x(k) - x^T(k - d_M) Q_2 x(k - d_M) \\
&\quad + x^T(k - d_m)(Q_2 - Q_1) x(k - d_m), \\
\Delta V_3(k) &= d_m^2 \eta^T(k) R_1 \eta(k) - d_m \sum_{i=k-d_m}^{k-1} \eta^T(i) R_1 \eta(i), \\
\Delta V_4(k) &= \rho^2 \eta^T(k) R_2 \eta(k) - \rho \sum_{i=k-d_M}^{k-1} \eta^T(i) R_2 \eta(i).
\end{align*}
\]
From (2), we have
\[
\psi_1(k) := (A - I)x(k) + A_d x(k - d(k)) - \eta(k) \equiv 0.
\]
Denote
\[
\alpha(k) := \sum_{i=k-d_m}^{k-1} \eta(i), \quad \beta(k) := \sum_{i=k-d_M}^{k-1} \eta(i)
\]
and
\[
\gamma(k) := \sum_{i=k-d_M}^{k-1} \eta(i).
\]
According to (1), we have \( \alpha(k) = 0 \) when \( d_m = 0 \), \( \beta(k) = 0 \) when \( d_M = 0 \), and \( \gamma(k) = 0 \) when \( d(k) = d_M \). Then, it is obvious that
\[
\begin{align*}
\psi_2(k) &= x(k) - x(k - d_m) - \alpha(k) \equiv 0, \\
\psi_3(k) &= x(k - d_m) - x(k - d(k)) - \beta(k) \equiv 0, \\
\psi_4(k) &= x(k - d(k)) - x(k - d_M) - \gamma(k) \equiv 0.
\end{align*}
\]
Denote \( \xi^T(k) = [\xi_1^T(k) \xi_2^T(k) \xi_3^T(k)] \), where
\[
\begin{align*}
\xi_1^T(k) &= [x^T(k) \; x^T(k - d_M)) \; \eta^T(k)], \\
\xi_2^T(k) &= [x^T(k - d_m) \; x^T(k - d_M)], \\
\xi_3^T(k) &= [\alpha^T(k) \; \beta^T(k) \; \gamma^T(k)].
\end{align*}
\]
Then, we have
\[
\begin{align*}
\Delta V_1(k) &= \eta^T(k) P \eta(k) + \eta^T(k) P x(k) + x^T(k) P \eta(k) \\
&\quad + 2\xi_1^T(k) T \psi_1(k) + 2\xi_2^T(k) L \psi_2(k) \\
&\quad + 2\xi_3^T(k) M \psi_3(k) + 2\xi_3^T(k) N \psi_4(k) \\
&= \xi^T(k) \Theta \xi(k),
\end{align*}
\]
where \( T, L, M \) and \( N \) are free weighting matrices,
\[
\begin{bmatrix}
\hat{\Omega} & M - L & -N & -L & -M & -N \\
* & Q_2 - Q_1 & 0 & 0 & 0 & 0 \\
* & * & -Q_2 & 0 & 0 & 0 \\
* & * & * & -R_1 & 0 & 0 \\
* & * & * & * & -R_2 & 0 \\
* & * & * & * & * & -R_2
\end{bmatrix},
\]
with
\[
\begin{bmatrix}
\Omega_{11} & \Omega_{12} & \Omega_{13} \\
* & \Omega_{22} & \Omega_{23} \\
* & * & \Omega_{33}
\end{bmatrix},
\]
and \( \Omega_{12}, \; \Omega_{13}, \; \Omega_{22}, \; \Omega_{23} \) are defined in (7). On the other hand, by applying Lemma 1, we can obtain
\[
\Delta V_3(k) \leq d_m^2 \eta^T(k) R_1 \eta(k) - \alpha^T(k) R_4 \alpha(k),
\]
(11)
Moreover, subject to polytopic uncertainty (4) is robustly stable if there exist matrices $P^{(i)} > 0$, $Q^{(i)}_1 > 0$, $Q^{(i)}_2 > 0$, $R^{(i)}_1 > 0$, $R^{(i)}_2 > 0$, $(i = 1, 2, \ldots, r)$, $T_1$, $T_2$, $T_3$, $L^T = [L^T_1 \ L^T_2 \ L^T_3]$, $M^T = [M^T_1 \ M^T_2 \ M^T_3]$ and $N^T = [N^T_1 \ N^T_2 \ N^T_3]$ of appropriate dimensions such that the following LMIs:

$$
\Theta_i < 0,
$$

hold for $i = 1, 2, \ldots, r$, where $\Theta_i$ is given by

$$
\hat{\Omega} = \begin{bmatrix}
\hat{\Omega}_{11} & \hat{\Omega}_{12} & \hat{\Omega}_{13} \\
* & \hat{\Omega}_{22} & \hat{\Omega}_{23} \\
* & * & \hat{\Omega}_{33}
\end{bmatrix},
$$

with $\rho = d_M - d_m$.

Remark 1. In the proof of Theorem 1, we introduce a new Lyapunov functional (8) to derive the stability criterion. Compared with those in Zhang et al. (2008) and Gao and Chen (2007), this functional is simpler. More precisely, the following terms:

$$
\Delta V_2(k) = -d_M \sum_{i = k - d_M}^{k-1} \sum_{i = k - d(k)}^{k-1} \eta^T(i) R_2 \eta(i)
$$

and

$$
\Delta V_6(k) = -d_m \sum_{j = -d_M + 1}^{k} \sum_{i = k + j}^{k-1} x^T(i) Q_3 x(i)
$$

are employed in Zhang et al. (2008), where $Q_3 > 0$. Moreover, $V_2(k)$ and $V_6(k)$ in (8) are a little different from those in Zhang et al. (2008). However, as indicated in Example 1, Theorem 1 can provide less conservative results than those in Zhang et al. (2008) and Gao and Chen (2007).

In what follows, on the basis of Theorem 1, we consider the robust stability of the system described by (2) and (3) subject to polytopic uncertainty (4).

Assume that the matrices $A$, $A_d$ in (2) have the form of (4). Then, based on Theorem 1, we can obtain the following result.

Theorem 2. The discrete-time delay system in (2) subject to polytopic uncertainty (4) is robustly stable for any time-varying delay $d(k)$ satisfying (3),
and \(P(s) > 0, Q_1^{(s)} > 0, Q_2^{(s)} > 0, R_1^{(s)} > 0, R_2^{(s)} > 0,\)
for given \(d_n\), we can easily deduce the LMI conditions (14) by following a similar line as that in the proof of Theorem 1.

IV. NUMERICAL EXAMPLES

In this section, we provide two numerical examples to show the comparison between several existing stability criteria proposed in recent literature and the results obtained in this paper. The first example is borrowed from Zhang et al. (2008).

**Example 1.** Consider the system (2) with
\[
A = \begin{bmatrix}
0.8 & 0 \\
0.05 & 0.9 
\end{bmatrix}, \quad A_d = \begin{bmatrix}
-0.1 & 0 \\
-0.2 & -0.1 
\end{bmatrix}.
\]

For given \(d_n\), we calculate the allowable maximum value of \(d_M\) that guarantees the asymptotic stability of system (2). By using different methods, the calculated results are presented in Table 1. From the table, we can see that Theorem 1 in this paper provides the least conservative result.

**Example 2.** Consider the following discrete system described by (2) and (3) subject to polytopic-type uncertainty (4) with
\[
A^{(1)} = \begin{bmatrix}
0.80 & 0 \\
0.01 & 0.6 
\end{bmatrix}, \quad A^{(2)} = \begin{bmatrix}
0.90 & 0 \\
0.05 & 0.9 
\end{bmatrix},
\]
\[
A_d^{(1)} = \begin{bmatrix}
0.1 & 0 \\
0.2 & 0.1 
\end{bmatrix}, \quad A_d^{(2)} = \begin{bmatrix}
-0.1 & 0 \\
-0.2 & -0.1 
\end{bmatrix}.
\]

For given \(d_n\), we compute the maximum allowable value of \(d_M\) such that the polytopic system is robustly asymptotically stable. Table 2 shows the computation results by using the stability criteria given in (Liu et al., 2006) and Theorem 2 in this paper. From the table, it is easy to see that the stability criterion obtained in this paper gives a much less conservative result than those in (Liu et al., 2006). In fact, the stability criterion proposed in (Liu et al., 2006) for polytopic-type systems is actually not a delay-dependent stability condition since it only depends on the size of the interval, i.e., \(d_M - d_n\).

V. CONCLUSION

We have addressed the stability problem of linear discrete-time systems with interval-like time-varying delay and polytopic uncertainty. A new Lyapunov-Krasovskii functional, which includes the range information of the delay, is proposed to derive a new delay-range-dependent stability criterion. The advantage of the proposed criterion lies in its less conservativeness. Robust stability condition has also been established for systems with polytopic-type uncertainty. Numerical examples show that the proposed criteria can provide less conservative results than some existing ones.

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