BEHAVIOR OF THE SOLUTION OF THE TWO-PHASE STEFAN PROBLEM WITH REGARD TO THE CHANGING OF THERMAL COEFFICIENTS OF THE SUBSTANCE

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***Abstract*— We consider one-dimensional two-phase Stefan problems for a finite substance with different boundary conditions at the fixed faces. The goal of this paper is to determine the behavior of the free boundary and the temperature when the thermal coefficients of the material change.**

We obtain properties of monotony with respect to the latent heat, the common mass density, the specific heat of each phase and the thermal conductivity of the liquid phase.

We show that the solution is not monotone with respect to the thermal conductivity of solid phase, in some cases, by computing a numerical solution through a finite difference scheme.

The results obtained are important in technological applications as the climate of buildings, the storage of energy in satellites and clothes and the transport of biological substances and telecommunications.

***Keywords*— Phase change material, Two phase Stefan problem, Finite difference method**

I. INTRODUCTION

Several technological applications can be modeled through heat transfer problems with phase-change. The Phase Change Materials (PCMs) are substances whose phase-change temperature make them available to moderate oscillations of temperature and to store energy of another substance in contact with them (Jiji and Gaye, 2006). For this reason PCMs are used for the climate of buildings, the storage of energy in satellites and clothes and the transport of biological substances (Asako *et al.*, 2002; Lamberg and Sirn, 2003), among a variety of applications.

Usually, the technological way to select a PCM is through its phase-change temperature, but when there are several products in the correct range of temperature, it is necessary to study another properties of the substance, for example the thermal coefficients, in order to choose the most convenient PCM.

When we consider a packaging of a PCM that recovers an organic substance to be transported, it is essential to find the thickness of this pack to insure the optimal temperature of conservation in the organic substance during total time of transport (Bouciguez *et al.*, 2001; Farid *et al.*, 2004; Medina *et al.*, 2004; Zalba *et al.*, 2003; Zivkovic and Fujii, 2001). Because the sizes of the pack (wide, length and height) are sufficiently greater than its thickness, we can assume that the heat transfer occurs in only one direction. Moreover, if we consider a material with one portion at solid state and the other at the liquid state, and take into account the environment conditions, we have an one dimensional two-phase Stefan problem (Lamberg, 2004; Lamberg *et al.*, 2004).

In Olguín *et al.* (2007), we considered a one-dimensional one-phase Stefan problem for fusion of a semi-infinite material. We showed that, both, the temperature and the free boundary present a monotonous behavior with respect to the latent heat, the mass density and the specific heat. We also showed that the solution is not monotone with respect to the thermal conductivity.

At the present work we continue this line of research. We consider several two-phase one-dimensional Stefan problems for a finite material, with different boundary conditions at the two fixed faces, in order to determine the behavior of the solutions when the thermal coefficients change.

In Section 2 we study a phase-change problem with temperature specification on both fixed faces of the finite material and we obtain results of monotony for latent heat, mass density, specific heat of the solid and the liquid phase, and thermal conductivity of the liquid phase, by using the maximum principle (Protter and Weinberger, 1967).

Analogous results are obtained in Section 3, by considering heat flux specification at both fixed faces of the finite material and, in Section 4, with temperature specification at the left fixed face and a heat flux specification at the right fixed face.

When we consider the thermal conductivity of the solid phase, it was not possible to establish analytical results of monotony. In order to obtain some conclusion for this case, in Section 5 we consider a numerical solution. Since the free boundary change with time, the domain of the problems is variable, then it is necessary to develop a particular scheme with a time variable mesh. We can show that, in some cases, the solutions are not monotone.

II. PROBLEM WITH TEMPERATURE BOUNDARY SPECIFICATION AT BOTH FACES

A. Mathematical problem and preliminary results

We consider a finite material represented by the interval $[0,1]$. At the initial time, we suppose that one portion of the material is at solid phase and the other is at liquid phase. If we consider that $s(t)$ is the position of the free boundary at each time and

$$w(x,t) = \begin{cases} u(x,t) & 0 \le x \le s(t),\ 0 \le t \le T, \\ v(x,t) & s(t) \le x \le 1,\ 0 \le t \le T, \end{cases}$$

is the temperature of the material, the problem $(P1)$ is to find functions $w(x,t)$, $s(t)$ and a time $T > 0$, so that they satisfy the following conditions:

$$\alpha_l\, u_{xx} = u_t, \quad 0 < x < s(t), 0 < t < T, \tag{1}$$

$$\alpha_s\, v_{xx} = v_t, \quad s(t) < x < 1, 0 < t < T, \tag{2}$$

$$u(0,t) = f(t) > 0, \quad 0 < t < T, \tag{3}$$

$$v(1,t) = g(t) < 0, \quad 0 < t < T, \tag{4}$$

$$u(s(t),t) = v(s(t),t) = 0, \quad 0 < t < T, \tag{5}$$

$$k_s v_x(s(t),t) - k_l u_x(s(t),t) = \rho \ell s'(t), \quad 0 < t < T, \tag{6}$$

$$u(x,0) = \varphi(x) \ge 0, \quad 0 \le x \le b, \tag{7}$$

$$v(x,0) = \psi(x) \le 0, \quad b \le x \le 1, \tag{8}$$

$$s(0) = b, \quad 0 < b < 1, \tag{9}$$

where $\alpha_i = \frac{k_i}{\rho\, c_i}$ is the thermal diffusivity of the phase i $(i = s, l)$ and $\varphi(b) = 0 = \psi(b)$ (see Fig.1).

In Cannon and Primicerio (1971), Tarzia (1987) and Cannon (1984), under suitable hypothesis for data, it was proved that:

$*$) there is a unique solution for problem $(P1)$ for all $T > 0$;

$**$) the Stefan condition $(Eq.6)$ is equivalent to the integral equation:

$$s(t) = b + \int_0^1 \Phi(x)dx - \frac{c_l}{\ell}\int_0^{s(t)} u(x,t)dx$$

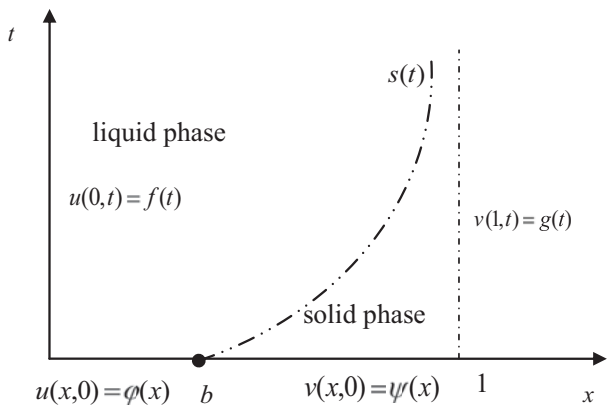

Figure 1: Scheme of the problem (P1)

$$-\frac{c_s}{\ell}\int_{s(t)}^1 v(x,t)dx + \frac{k_s}{\rho\ell}\int_0^t v_x(1,\tau)d\tau$$

$$-\frac{k_l}{\rho\ell}\int_0^t u_x(0,\tau)d\tau\, t \ge 0 \tag{10}$$

where

$$\Phi(x) = \begin{cases} \frac{c_l}{\ell}\,\varphi(x), & 0 \le x \le b, \\ \frac{c_s}{\ell}\,\psi(x), & b < x \le 1. \end{cases}$$

B. Analytical results of monotony

In this Section we use the maximum principle, Hopf's lemma (Protter and Weinberger, 1967; Cannon, 1984) and the result of the following Lemma, when it is required, in order to establish some properties of monotony for the solution of problem $(P1)$.

Lemma 1 *Let w be the solution of $(P1)$. Then:*

i) $u_x(s(t),t) < 0$ and $v_x(s(t),t) < 0$ for $t > 0$;

ii) If the function $f(t)$ is continuously differentiable for $t > 0$ and $\varphi(x)$ is twice continuously differentiable at $0 < x < b$, $f'(t) \le 0$ and $\varphi''(x) \ge 0$ then $u_{xx}(x,t) \ge 0$ for $0 < x < s(t)$, $0 < t < T$.

iii) If $g(t)$ is continuously differentiable for $t > 0$ and $\psi(x)$ is twice continuously differentiable at $b < x < 1$, $g'(t) \ge 0$ and $\psi''(x) \ge 0$ then $v_{xx}(x,t) \ge 0$ for $s(t) < x < 1$, $0 < t < T$.

Proof.

i) The results are obtained when the maximum principle and Hopf's lemma are applied to functions u and v respectively.

ii) We define the auxiliary function $z = u_{xx}$ which verify the associated problem

$$\begin{array}{ll} z_t = \alpha_l\, z_{xx}, & 0 < x < s(t),\ t > 0, \\ z(0,t) = \frac{1}{\alpha_l}\, f'(t) & t > 0, \\ z(x,0) = \varphi''(x) & 0 < x < b, \\ z(s(t),t) \ge 0, & t > 0, \\ s(0) = b. & \end{array}$$

By using the maximum principle, we obtain $z \ge 0$.

iii) The proof follows (ii) by defining now the auxiliary function $z = v_{xx}$.

Proposition 2 *The solution $\{w(x,t),\ s(t)\}$ of problem $(P1)$ depends monotonically on the latent heat ℓ, the mass density ρ, the specific heat $c_i\,(i=s,l)$ and the thermal conductivity of the liquid phase k_l.*

Proof. *The five results are similar but the proofs are different because we must change the coefficients only in the Stefan condition for the latent heat ℓ, only in the heat equation for the specific heats $c_i(i=s,l)$ and in both equations in the remaining cases.*

a) We consider $\ell_1<\ell_2$. We call $\{w_1,\ s_1\}$, $\{w_2,\ s_2\}$ the solutions of problem $(P1)$ with $\ell=\ell_i$ and $s_i(0)=b_i$ for $i=1,2$. We consider two cases: $s_1(0)>s_2(0)$ and $s_1(0)\geq s_2(0)$.

***Case** I: Let $b_1=s_1(0)>s_2(0)=b_2$. We suppose t_0 is the first time such that $s_1(t_0)=s_2(t_0)$ and $s_2(t)<s_1(t)$ for all $0<t<t_0$. Because of that, it must occur (Fig.2):*

$$s_1^{'}(t_0)\ \leq\ s_2^{'}(t_0) \tag{11}$$

If we define the functions

$$\begin{cases} U(x,t)=u_1(x,t)-u_2(x,t),\ 0\leq x\leq s_2(t),\\ V(x,t)=v_1(x,t)-v_2(x,t),\ s_1(t)\leq x\leq 1,\end{cases}$$

for $0\leq t\leq t_0$, then U satisfies the following conditions:

$$U_t-\alpha_l\,U_{x,x}=0,\quad 0<x<s_2(t),\quad 0<t<t_0, \tag{12}$$

$$U(0,t)=0,\quad 0<t<t_0, \tag{13}$$

$$U(x,0)=0\quad 0\leq x\leq b_2, \tag{14}$$

$$U(s_2(t),t)=u_1(s_2(t),t)>0,\quad 0<t<t_0, \tag{15}$$

$$s_2(0)=\ b_2. \tag{16}$$

By the minimum principle, condition $(Eq.15)$ holds and then $U\geq 0$ in $0\leq x\leq s_2(t)$, $0\ \leq\ t\ \leq\ t_0$. At the point $x=s_1(t_0)=s_2(t_0)$,we have $U(s_2(t_0),t_0)=0$, therefore U attains a minimum at $(s_2(t_0),t_0)$. By the Hopf's lemma, we have

$$U_x\,(s_2(t_0),t_0)<0. \tag{17}$$

On the other hand, the function V, satisfies the following conditions:

$$V_t-\ \alpha_s V_{xx}=0,\quad s_1(t)<x<1,\quad 0<t<t_0, \tag{18}$$

$$V(s_1(t),t)=-v_2(s_1(t),t)>0,\quad 0<t<t_0, \tag{19}$$

$$V(x,0)=0,\quad b_1\leq x\leq 1, \tag{20}$$

$$V(1,t)=0,\quad 0<t<t_0, \tag{21}$$

$$s_1(0)=b_1. \tag{22}$$

In a similar way as above, we obtain that $V(x,t)\geq 0$ in $s_1(t)\ \leq\ x\ \leq\ 1$, $0\ \leq\ t\ \leq\ t_0$. Taking into account that $V(s_1(t_0),t_0)\ =\ 0$, then V has a minimum at $(s_1(t_0),t_0)$ and from the Hopf's lemma, it results

$$V_x\,(s_1(t_0),t_0)>0. \tag{23}$$

But, from the Stefan condition $(Eq.6),(Eq.11)$ and $(Eq.17)$ we obtain:

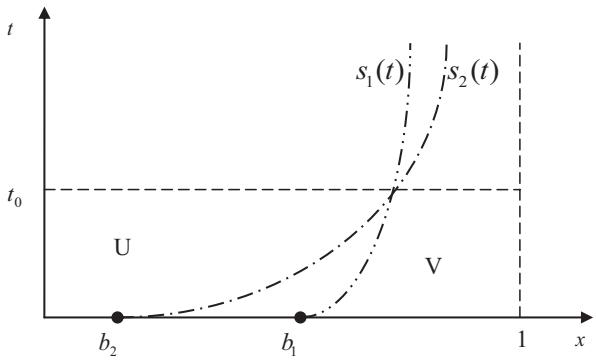

Figure 2: Free boundaries for $\ell_1<\ell_2$

$$V_x(s_1(t_0),t_0)=v_{1x}(s_1(t_0),t_0)-v_{2x}(s_2(t_0),t_0)$$
$$=\tfrac{\rho}{k_s}\,[\ell_1 s_1^{'}(t_0)-\ \ell_2\,s_2^{'}(t_0)]+\tfrac{k_l}{k_s}\,U_x(s_2(t_0),t_0)$$
$$<\tfrac{\rho\,\ell_1}{k_s}\,[s_1^{'}(t_0)-s_2^{'}(t_0)]+\tfrac{k_l}{k_s}U_x(s_2(t_0),t_0)<0,$$

which is a contradiction with $(Eq.23)$. Then we conclude that:

$$\begin{cases} s_1(t)>s_2(t),\\ u_1(x,t)\geq u_2(x,t),\quad 0\leq x\leq s_2(t),\\ v_1(x,t)\geq v_2(x,t),\quad s_1(t)\leq x\leq 1\end{cases}$$

for $0\leq t\leq T$.

***Case** II:We consider now $b_1=s_1(0)\geq s_2(0)=b_2$. Let $\delta\ >\ 0$,then $b_2\ \leq\ b_1\ <\ b_1+\delta$. Let w_δ,s_δ be the solution of $(P1)$ whit latent heat ℓ_1 and $s_\delta(0)=b_1+\delta$ in $(Eq.9)$. We define the functions φ_δ and ψ_δ on the intervals$[0,b_1+\delta]$ and $[b_1+\delta,1]$respectively by:*

$$\varphi_\delta(x)=\varphi\Big(\frac{b_1\,x}{b_1+\delta}\Big)$$

and

$$\psi_\delta(x)=\psi\Big(\frac{(1-b_1)\,x-\delta}{1-b_1-\delta}\Big).$$

From case I, we have that:

$$s_2(t)<s_\delta(t)\ \forall\ t\geq 0.$$

Taking into account the Stefan condition $(Eq.10)$ for s_1 and s_δ, and from Case I, it results that:

$$[s_\delta(t)-s_1(t)]=\delta+\tfrac{1}{\rho\ell_1}\int_0^t k_s[v_{\delta x}(1,\tau)-v_{1x}(1,\tau)]d\tau-$$
$$\tfrac{1}{\rho\,\ell_1}\int_0^t\ k_l[u_{\delta x}(0,\tau)-u_{1x}(0,\tau)]\,d\tau$$
$$-\tfrac{c_l}{\ell_1}\Big[\int_0^{s_\delta(t)}u_\delta(x,t)\,dx\quad-\quad\int_0^{s_1(t)}u_1(x,t)\,dx\Big]\quad+$$
$$\tfrac{c_s}{\ell_1}\Big[\int_{s_1(t)}^1 v_1(x,t)\,dx-\int_{s_\delta(t)}^1 v_\delta(x,t)\,dx\Big]$$
$$+\tfrac{c_l}{\ell_1}\Big[\int_0^{b_1+\delta}u_\delta(x,0)\,dx\quad-\quad\int_0^{b_1}u_1(x,0)\,dx\Big]\quad+$$
$$\tfrac{c_s}{\ell_1}\Big[-\int_{b_1}^1 v_1(x,0)\,dx+\int_{b_1+\delta}^1 v_\delta(x,0)\,dx\Big]$$
$$<\quad\delta\ +\ \tfrac{1}{\rho\,\ell_1}\int_0^t k_s\Big[v_{\delta x}(1,\tau)\ -\ v_{1x}(1,\tau)\Big]d\tau\ -$$
$$\tfrac{1}{\rho\ell_1}\int_0^t k_l\Big[u_{\delta x}(0,\tau)-u_{1x}(0,\tau)\Big]d\tau$$
$$+\tfrac{c_l}{\ell_1}\Big[\int_0^{b_1}\Big[\varphi_\delta(x)\ -\ \varphi(x)\Big]dx\ +\ \int_{b_1}^{b_1+\delta}\varphi_\delta(x)\,dx\Big]\ +$$
$$\tfrac{c_s}{\ell_1}\Big[\int_{b_1+\delta}^1\Big[\psi_\delta(x)-\psi(x)\Big]dx\ -\int_{b_1}^{b_1+\delta}\psi(x)\,dx\Big].$$

If we apply again the maximum principle and Hopf's lemma, we can see that:
$v_{\delta x}(1,\tau) - v_{1x}(1,\tau) < 0$ *and* $u_{\delta x}(0,\tau) - u_{1x}(0,\tau) > 0$, *then*
$s_\delta(t) - s_1(t) < \delta + \frac{c_l}{\ell_1}\Big[\int_0^{b_1}\Big[\varphi_\delta(x) - \varphi(x)\Big]dx + \int_{b_1}^{b_1+\delta}\varphi_\delta(x)dx\Big] + \frac{c_s}{\ell_1}\Big[\int_{b_1+\delta}^{1}\Big[\psi_\delta(x) - \psi(x)\Big]dx + \int_{b_1}^{b_1+\delta}\psi(x)\,dx\Big], \quad \forall\, \delta > 0,$
which implies
$s_2(t) < s_\delta(t) < s_1(t) + \delta + \frac{c_l}{\ell_1}\int_0^{b_1}\Big[\varphi_\delta(x) - \varphi(x)\Big]dx + \frac{c_l}{\ell_1}\int_{b_1}^{b_1+\delta}\varphi_\delta(x)\,dx - \frac{c_s}{\ell_1}\int_{b_1+\delta}^{1}\Big[\psi_\delta(x) - \psi(x)\Big]dx$
$-\frac{c_s}{\ell_1}\int_{b_1}^{b_1+\delta}\psi(x)dx, \quad \forall\, \delta > 0.$

Taking the limit when $\delta \to 0$, *we have that* $s_2 \le s_1$ *in the common domain of existence. To complete the proof, we consider for* $i = 1, 2$

$$w_i(x,t) = \begin{cases} u_i(x,t) & 0 \le x \le s_i(t), \quad 0 < t < T; \\ v_i(x,t) & s_i(t) \le x \le 1, \quad 0 < t < T. \end{cases}$$

As we have seen before, $w_1 - w_2 \ge 0$ *for* $0 \le x \le s_2(t)$ *and* $s_1(t) \le x \le 1$. *For* $s_2(t) \le x \le s_1(t)$, *we have* $w_1(x,t) - w_2(x,t) = u_1(x,t) - v_2(x,t) \ge 0$, *because* $u_1 \ge 0$ *and* $v_2 \le 0$ *and the thesis holds.*
b) Now, we consider that the thermal conductivity of the liquid phase changes. Let $k_{l1} < k_{l2}$ *and we note as* $\{w_1,\ s_1\}$, $\{w_2,\ s_2\}$ *the solutions of problem* $(P1)$ *with* $k_l = k_{li}$ *and* $s_i(0) = b_i$ *for* $i = 1, 2$. *As before, we consider two cases:* $s_1(0) > s_2(0)$ *and* $s_1(0) \ge s_2(0)$.

***Case** I:* $b_1 = s_1(0) < s_2(0) = b_2$ *and* t_0 *is the first time such that* $s_1(t_0) = s_2(t_0)$, *then we have (see Fig.3):*

$$s_1'(t_0) \ge s_2'(t_0). \tag{24}$$

If we define the function

$$\begin{cases} U(x,t) = u_2(x,t) - u_1(x,t), & 0 \le x \le s_1(t), \quad t \ge 0, \\ V(x,t) = v_2(x,t) - v_1(x,t), & s_2(t) \le x \le 1, \quad t \ge 0, \end{cases}$$

we see that U *satisfies the following conditions:*

$$U_t - \frac{k_{l2}}{\rho c_l}U_{xx} \ge 0,\ 0 < x < s_1(t), \quad 0 < t < t_0, \tag{25}$$

$$U(0,t) = 0, \quad 0 < t < t_0, \tag{26}$$

$$U(x,0) = 0, \quad 0 \le x \le b_1, \tag{27}$$

$$U(s_1(t),t) = u_2(s_1(t),t) - 0 > 0, \quad 0 < t < t_0, \tag{28}$$

$$s_1(0) = b_1. \tag{29}$$

The inequality (Eq.25) is a consequence of part (ii) of Lemma 1, while the condition (Eq.28) is obtained from the minimum principle and then we conclude that $U \ge 0$ *in* $0 \le x \le s_1(t), 0 \le t \le t_0$. *At* $x = s_1(t_0) = s_2(t_0)$, *we have* $U(s_1(t_0),t_0) = 0$, *then* U *has a minimum at* $(s_1(t_0),t_0)$ *and from the Hopf's lemma, it results*

$$U_x(s_1(t_0),t_0) < 0. \tag{30}$$

On the other hand, the function V *satisfies the following conditions:*

$$V_t - \alpha_s V_{xx} = 0,\ s_2(t) < x < 1, \quad 0 < t < t_0, \tag{31}$$

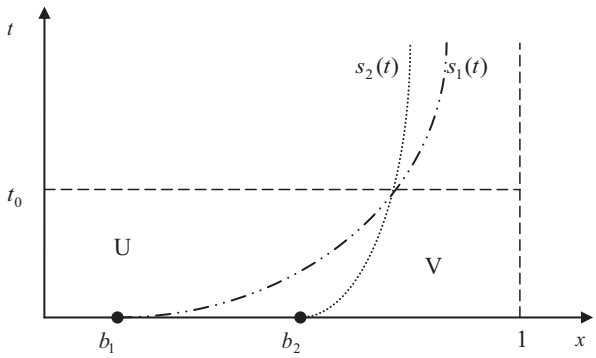

Figure 3: Free boundaries for $k_{l1} < k_{l2}$

$$V(s_2(t),t) = -v_1(s_2(t),t) > 0, \quad 0 < t < t_0, \tag{32}$$

$$V(x,0) = 0, \quad b_2 \le x \le 1, \tag{33}$$

$$V(1,t) = 0, \quad 0 < t < t_0, \tag{34}$$

$$s_2(0) = b_2. \tag{35}$$

In the same way as above, we have that $V(x,t) \ge 0$ *in* $s_2(t) \le x \le 1, \quad 0 \le t \le t_0$. *Since* $V(s_2(t_0),t_0) = 0$, V *has a minimum at* $(s_2(t_0),t_0)$*and from Hopf's lemma, it results*

$$V_x(s_2(t_0),t_0) > 0. \tag{36}$$

Therefore, from the Stefan condition (Eq.6), (Eq.24), (Eq.30) and Lemma 1 (i) we have:
$V_x(s_1(t_0),t_0) = v_{2x}(s_1(t_0),t_0) - v_{1x}(s_1(t_0),t_0)$
$= \frac{\rho\ell}{k_s}[s_2'(t_0) - s_1'(t_0)] + \Big[\frac{k_{l2}}{k_s}U_x(s_1(t_0),t_0)$
$+ \frac{k_{l2}-k_{l1}}{k_s}u_{1x}(s_1(t_0),t_0)\Big] < 0,$
which is a contradiction with (Eq.36). Then we conclude that:

$$\begin{cases} s_1(t) < s_2(t), \\ u_1(x,t) \le u_2(x,t), & 0 \le x \le s_1(t), \\ v_1(x,t) \le v_2(x,t), & s_2(t) \le x \le 1 \end{cases}$$

for $0 \le t \le T$.

***Case** II:* *Let* $b_1 = s_1(0) \le s_2(0) = b_2$. *Let* $\delta > 0$, *then* $b_1 \le b_2 < b_2 + \delta$. *We note* $\{w_\delta, s_\delta\}$ *the solution of the problem* $(P1)$ *with thermal conductivity of liquid phase* k_{l2} *and* $s_\delta(0) = b_2 + \delta$. *We define the functions* φ_δ *and* ψ_δ *on the intervals* $\big[0, b_2+\delta\big]$ *and* $\big[b_2+\delta, 1\big]$ *respectively by:*
$\varphi_\delta(x) = \varphi\Big(\frac{b_2\,x}{b_2+\delta}\Big)$ *and* $\psi_\delta(x) = \psi\Big(\frac{(1-b_2)\,x-\delta}{1-b_2-\delta}\Big)$.
From case I, we have that:

$$s_1(t) < s_\delta(t) \quad \forall\, t \ge 0.$$

From the Stefan integral condition (Eq.10) for s_2 *and* s_δ *and Case I we obtain:*
$[s_\delta(t) - s_2(t)] = \delta + \frac{1}{\rho\ell}k_s\int_0^t\Big[v_{\delta x}(1,\tau) - v_{2x}(1,\tau)\Big]d\tau +$
$\frac{1}{\rho\ell}k_{l2}\int_0^t\Big[u_{2x}(0,\tau) - u_{\delta x}(0,\tau)\Big]d\tau - \frac{c_l}{\ell}\int_0^{s_2(t)}\Big[u_\delta(x,t)$
$-u_2(x,t)\Big]dx - \frac{c_l}{\ell}\int_{s_2(t)}^{s_\delta(t)}u_\delta(x,t)dx + \frac{c_s}{\ell}\int_{s_\delta(t)}^{1}\Big[v_2(x,t)$

$-v_\delta(x,t)\Big]dx+\frac{c_l}{\ell}\int_{s_2(t)}^{s_\delta(t)} v_2(x,t)dx+\frac{c_l}{\ell}\Big[\int_0^{b_2+\delta}\varphi_\delta(x)dx$
$-\int_0^{b_2}\varphi(x)dx\Big]+\frac{c_s}{\ell}\Big[\int_{b_2+\delta}^1\psi_\delta(x)\,dx-\int_{b_2}^1\psi(x)dx\Big].$

If we apply the maximum principle and Hopf's lemma, we obtain that

$\Big[v_{\delta x}(1,\tau)-v_{2x}(1,\tau)\Big]<0$ *and*

$\Big[u_{2x}(0,\tau)-u_{\delta x}(0,\tau)\Big]<0$

and consequently:

$s_\delta(t)-s_2(t)<$
$\delta+\frac{c_l}{\ell}\Big[\int_0^{b_2+\delta}\varphi_\delta(x)dx-\int_0^{b_2}\varphi(x)dx\Big]+\frac{c_s}{\ell}\Big[\int_{b_2+\delta}^1\psi_\delta(x)\,dx-\int_{b_2}^1\psi(x)\,dx\Big],\quad \forall\,\delta>0.$

When δ tends to zero, we obtain that $s_1(t)\le s_2(t)$ in the common domain of existence.

c) By a similar argument as above, applying the maximum principle and the Hopf's lemma when it is necessary, we obtain the thesis taking into account that in this case the specific heat of each phase changes. We must consider that in this situation the heat equations (Eq.1) or (Eq.2) are affected.

d) As in c), when the mass density changes, we can show the monotony considering that the heat equations (Eq.1), (Eq.2) and also the Stefan condition (Eq.6) are affected.

III. PROBLEM WITH FLUX BOUNDARY SPECIFICATION AT BOTH FACES

A. Mathematical problem and preliminary results

We consider a similar problem to $(P1)$, but on $x=0$ and on $x=1$ the conditions $(Eq.3)$ and $(Eq.4)$ are replaced by heat flux conditions respectively. Then, we define the problem $(P2)$ for equations $(Eq.1),(Eq.2),(Eq.5)-(Eq.9)$ and

$$k_l u_x(0,t)=f(t)\le 0,\ 0<t<T, \tag{37}$$

$$k_s v_x(1,t)=g(t)\le 0,\ 0<t<T. \tag{38}$$

In Tarzia (1987), Cannon (1971) and Cannon (1984) under certain hypotheses, it was proved that: *) there is unique solution for problem $(P2)$ in $0<t<T_0$, for some $T_0>0$;

**) the condition $(Eq.6)$ is equivalent to the integral equation for all $t\ge 0$:

$$s(t)=b+\int_0^1\Phi(x)\,dx-\frac{c_l}{\ell}\int_0^{s(t)}u(x,t)dx-$$

$$\frac{c_s}{\ell}\int_{s(t)}^1 v(x,t)\,dx+\frac{1}{\rho\,\ell}\int_0^t\Big[g(\tau)-f(\tau)\Big]d\tau, \tag{39}$$

where $\Phi(x)=\begin{cases}\frac{c_l}{\ell}\,\varphi(x) & 0\le x\le b,\\ \frac{c_s}{\ell}\,\psi\,(x) & b<x\le 1.\end{cases}$

B. Analytical results of monotony

Proposition 3 *Let $\{w(x,t),\ s(t)\}$ be the solution of $(P2)$. It depends monotonically on the latent heat ℓ, the mass density ρ, the specific heat c_i $(i=s,l)$ and the thermal conductivity of the liquid phase k_l.*

Proof.

a) When the latent heat ℓ changes, we consider two cases, as in the proof of Proposition 2:

***Case** I: $b_1=s_1(0)>s_2(0)=b_2$ and*

***Case** II: $b_1=s_1(0)\ge s_2(0)=b_2$.*

In both cases we repeat the proof but we consider, for the auxiliary functions U and V, the following flux data:

$$U_x(0,t)=0,\ 0<t<t_0,$$

$$V_x(1,t)=0,\ 0<t<t_0.$$

In addition, in Case II, as before, we consider the functions φ_δ and ψ_δ on the intervals $\big[0,b_1+\delta\big]$ and $\big[b_1+\delta,1\big]$ respectively by:

$\varphi_\delta(x)=\varphi\Big(\frac{b_1 x}{b_1+\delta}\Big)$ *and* $\psi_\delta(x)=\psi\Big(\frac{(1-b_1)\,x-\delta}{1-b_1-\delta}\Big)$.

From the integral Stefan condition (Eq.39) for s_1 and s_δ and considering Case I we have:

$[s_\delta(t)-s_1(t)]=$
$\delta+\frac{c_l}{\ell_1}\int_0^{b_1}[\varphi_\delta(x)-\varphi(x)]dx+\frac{c_l}{\ell_1}\int_{b_1}^{b_1+\delta}\varphi_\delta(x)dx+\frac{c_s}{\ell_1}\int_{b_1+\delta}^1[\psi_\delta(x)-\psi(x)]dx$
$-\frac{c_s}{\ell_1}\int_{b_1}^{b_1+\delta}\psi(x)dx-\frac{c_l}{\ell_1}\Big[\int_0^{s_1(t)}[u_\delta(x,t)-u_1(x,t)]dx+\int_{s_1(t)}^{s_\delta(t)}u_\delta(x,t)dx\Big]-\frac{c_s}{\ell_1}\Big[\int_{s_\delta(t)}^1[v_\delta(x,t)-v_1(x,t)]\,dx-\int_{s_1(t)}^{s_\delta(t)}v_1(x,t)dx\Big]\le\delta+\frac{c_l}{\ell_1}\int_0^{b_1}[\varphi_\delta(x)-\varphi(x)]dx+\frac{c_l}{\ell_1}\int_{b_1}^{b_1+\delta}\varphi_\delta(x)dx+\frac{c_s}{\ell_1}\int_{b_1+\delta}^1[\psi_\delta(x)-\psi(x)]dx$
$-\frac{c_s}{\ell_1}\int_{b_1}^{b_1+\delta}\psi(x)dx,\qquad\forall\delta>0,$

and consequently:

$s_2(t)<s_\delta(t)\le$
$s_1(t)+\delta+\frac{c_l}{\ell_1}\int_0^{b_1}[\varphi_\delta(x)-\varphi(x)]\,dx+\frac{c_l}{\ell_1}\int_{b_1}^{b_1+\delta}\varphi_\delta(x)dx$
$+\frac{c_s}{\ell_1}\int_{b_1+\delta}^1[\psi_\delta(x)-\psi(x)]dx-\frac{c_s}{\ell_1}\int_{b_1}^{b_1+\delta}\psi(x)dx,\ \forall\delta>0.$

Taking the limit as δ tends to 0, we obtain that $s_2\le s_1$ in the common domain of existence. In order to complete the proof, we continue as in a) of Proposition 2.

The proof of the Proposition 3 for the other parameters is analogous to the cases b), c) and d) of Proposition 2.

IV. PROBLEM WITH TEMPERATURE SPECIFICATION ON $x=0$ AND FLUX SPECIFICATION ON $x=1$

A. Mathematical problem and preliminary results

We consider a similar problem to $(P1)$, but on $x=0$ and $x=1$ the conditions $Eq.(3)$ and $Eq.(4)$ are

replaced by temperature specification and heat flux condition respectively. Then, we define the problem $(P3)$ for e $Eqs.(1),(2),(5-9)$ and

$$u(0,t) = f(t) > 0,\ 0 < t < T, \tag{40}$$

$$k_s v_x(1,t) = g(t) \leq 0,\ 0 < t < T. \tag{41}$$

In this case, under suitable hypothesis, it can be prove that the Stefan condition $(Eq.6)$ is equivalent to the following integral equation for all $t \geq 0$:

$$s(t) = b + \int_0^1 \Phi(x,0)dx - \frac{c_l}{\ell}\int_0^{s(t)} u(x,t)dx$$

$$-\frac{c_s}{\ell}\int_{s(t)}^1 v(x,t)dx + \frac{1}{\rho\ell}\int_0^t \Big[k_s g(\tau) - k_l u_x(0,\tau)\Big]d\tau, \tag{42}$$

where $\Phi(x) = \begin{cases} \frac{c_l}{\ell}\varphi(x),\ 0 \leq x \leq b, \\ \frac{c_s}{\ell}\psi(x)\ b < x \leq 1. \end{cases}$

B. Analytical results of monotony

Proposition 4 *Let $\{w(x,t), s(t)\}$ be the solution of $(P3)$. It depends monotonically on the latent heat ℓ, the mass density ρ, the specific heat $c_i (i = s, l)$ and the thermal conductivity of the liquid phase k_l.*

Proof.

a) We consider that the thermal conductivity of the liquid phase changes and, as in the proof of item b) of Proposition 2(b), we take into account two cases:

Case I: $b_1 = s_1(0) < s_2(0) = b_2$ *and*

Case II: $b_1 = s_1(0) \leq s_2(0) = b_2$.

We consider, for the auxiliary function V, a flux specification:

$$V_x(1,t) = 0,\ 0 < t < t_0,$$

and in Case II, as before, we consider the functions φ_δ and ψ_δ on the intervals$[0, b_2+\delta]$ and $[b_2+\delta, 1]$respectively by:

$\varphi_\delta(x) = \varphi\Big(\frac{b_2\, x}{b_2+\delta}\Big)$ *and*

$\psi_\delta(x) = \psi\Big(\frac{(1-b_2)x-\delta}{1-b_2-\delta}\Big)$.

From integral expression of the Stefan condition $(Eq.42)$ for s_2 and s_δ and Case I we have:

$[s_\delta(t) - s_2(t)] = \delta + \frac{c_l}{\ell}\int_0^{b_2}\Big[\varphi_\delta(x) - \varphi(x)\Big]dx + \frac{c_l}{\ell}\int_{b_2}^{b_2+\delta}\varphi_\delta(x)dx - \frac{c_s}{\ell}\int_{b_2}^{b_2+\delta}\psi(x)dx$
$+\frac{c_s}{\ell}\int_{b_2+\delta}^1\Big[\psi_\delta(x) - \psi(x)\Big]dx + \frac{1}{\rho\,\ell}k_{l2}\int_0^t\Big[u_{2x}(0,\tau) - u_{\delta x}(0,\tau)\Big]d\tau$
$-\frac{c_l}{\ell}\int_0^{s_2(t)}\Big[u_\delta(x,t) - u_2(x,t)\Big]dx - \frac{c_l}{\ell}\int_{s_2(t)}^{s_\delta(t)}u_\delta(x,t)dx$
$+\frac{c_s}{\ell}\int_{s_\delta(t)}^1\Big[v_2(x,t) - v_\delta(x,t)\Big]dx + \frac{c_s}{\ell}\int_{s_2(t)}^{s_\delta(t)}v_2(x,t)dx.$

By the maximum principle and the Hopf's lemma, we obtain

$$\Big[u_{2x}(0,\tau) - u_{\delta x}(0,\tau)\Big] < 0$$

which implies that

$s_\delta(t) - s_2(t) \leq \delta + \frac{c_l}{\ell}\int_0^{b_2}\Big[\varphi_\delta(x) - \varphi(x)\Big]dx + \frac{c_l}{\ell}\int_{b_2}^{b_2+\delta}\varphi_\delta(x)dx - \frac{c_s}{\ell}\int_{b_2}^{b_2+\delta}\psi(x)dx$
$+\frac{c_s}{\ell}\int_{b_2+\delta}^1\Big[\psi_\delta(x) - \psi(x)\Big]dx, \quad \forall\delta > 0,$
and consequently, if $\delta \to 0$, we obtain $s_1(t) \leq s_2(t)$ in the common domain of existence.

The proof for the other parameters is analogous to the Proposition 2.

Remark: We can not obtain any conclusion, through the maximum principle, about variations in the thermal conductivity k_s of the solid phase. This case will be studied by a numerical approximation in Section 5.

V. NUMERICAL SOLUTIONS AND MONOTONY

A. Numerical scheme

In this section we propose a numerical method in order to approximate the solution $\{w(x,t), s(t)\}$ of the problems $P(q)$, for $q = 1,2,3$. Since the free boundary change with time, the domain of the problem is variable. The numerical solution of free boundary problems of this type can be computed through different methods: front-traking methods and front-fixing methods (Zerroukat and Chatwin, 1994; Meyer, 1971; Crank, 1984). In this paper, we develop a scheme with a time variable mesh We propose a variable grid with fixed time step and with a constant number of space steps. These space steps will be update at each time level so that the free boundary is located on a node of the mesh (Fig. 4).

We consider a fixed time step Δt and:

$*$) $\mathrm{t}_{j+1} = j\Delta t$ for $j = 0,1,..,N$. (N number of time intervals so that $t_{N+1} \leq T$, time of coexistence of both phases).

$*$) For each time t_{j+1}, we define a partition of $[0,1]$, and we take m steps in each phase as follow:

- in the liquid phase:

$\Delta x_{lj} = \frac{s_j}{m}$, then $x_{ij} = (i-1)\Delta x_{lj}, i = 1,...,m+1$

- in the solid phase:

$\Delta x_{\mathrm{s}j} = \frac{1-s_j}{m}$, in consequence

$x_{ij} = s_j + (i-m-1)\Delta x_{\mathrm{s}j}, i = m+2,...,2m+1.$

where $s_j \simeq s(t_j)$.

$*$) $\mathrm{w}_{i,j+1} \simeq w(x_{ij}, t_{j+1}), , i = 1,..,2m+1,\ j = 0,..,N.$

$*$)$(x_{m+1,j}, t_{j+1}) \simeq (s_{j+1}, t_{j+1})$ for $j = 0,..,N$.

We define the coefficients:

$r_{l,j} = \alpha_l\frac{\Delta t}{(\Delta x_{l,j})^2}; \quad \beta_{l,i,j} = \frac{\Delta t\,(i-1)s'_j}{m\Delta x_{l,j}};$

$r_{\mathrm{s},j} = \alpha_{\mathrm{s}}\frac{\Delta t}{(\Delta x_{\mathrm{s},j})^2}; \quad \beta_{s,i,j} = \frac{\Delta t\,(2\mathrm{m}+1\text{-}i)\,s'_j}{1-s_j}$

where, we note $s'_j \simeq s'(t_j)$, and in each phase we approach the heat equation by an explicit finite difference scheme.

In order to approximate the distribution of temperature in $(0,1)$ x $(0,T)$ we replace equations $(Eq.1)$ and $(Eq.2)$ with $j = 0,..,N$ by:

$$\mathrm{w}_{i,j+1} = (r_{l,j} - \beta_{l,i,j})\mathrm{w}_{i+1,j}$$
$$+(1 - 2r_{l,j} + \beta_{l,i,j})\mathrm{w}_{i,j} + r_{l,j}\mathrm{w}_{i-1,j}; i = 2,..,m; \tag{43}$$

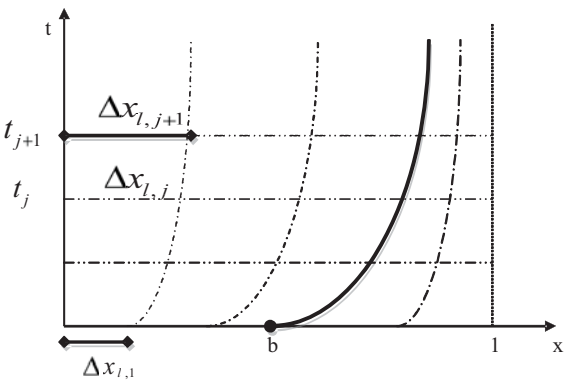

Figure 4: scheme for the time variable grid

$$\mathrm{w}_{i,j+1} = (r_{s,j} - \beta_{s,i,j})\mathrm{w}_{i+1,j}$$
$$+(1 - 2r_{s,j} + \beta_{s,i,j})\mathrm{w}_{i,j} + r_{s,j}\mathrm{w}_{i-1,j}; i = m+2,..,2m. \quad (44)$$

In an analogous way, we make the approximations of the boundary conditions:
for $(P1)$: $\mathrm{w}_{1,j+1} = f_{j+1}$;

$$\mathrm{w}_{2m+1,j+1} = g_{j+1}. \quad (45)$$

for $(P2)$: $\mathrm{w}_{1,j+1} = \mathrm{w}_{2,j+1} - \frac{f_{j+1}\Delta x_{l,j+1}}{k_l}$;

$$\mathrm{w}_{2m+1,j+1} = \mathrm{w}_{2m,j+1} + \frac{g_{j+1}\Delta x_{s,j+1}}{k_s}. \quad (46)$$

for $(P3)$: $\mathrm{w}_{1,j+1} = f_{j+1}$;

$$\mathrm{w}_{2m+1,j+1} = \mathrm{w}_{2m,j+1} + \frac{g_{j+1}\Delta x_{s,j+1}}{k_s}. \quad (47)$$

We obtain the numerical solution through the following algorithm:
Step 1:
$*$ Set the initial conditions and the boundary conditions
$*$ Choose $\Delta t > 0$ and m;
Step 2: Set $j = 1$
- compute $\Delta x_{l,j}$; $\Delta x_{s,j}$,
- evaluate s'_j from the Stefan condition $(Eq.6)$,
- $\mathrm{w}_{m+1,j+1} = 0$ (temperature on the free boundary),
- compute $\mathrm{w}_{i,j+1}$ $i = 2,..,m,m+2,...,2m$, from $(Eq.43)$ and $(Eq.44)$,
- compute $\mathrm{w}_{1,j+1}$, $\mathrm{w}_{2m+1,j+1}$from $(Eq.45),(Eq.46)$ or $(Eq.47)$ depending on the problem,

Step 3: While $j \leq N$ return to Step 2 with $j = j+1$;

Remark: Since we use an explicit scheme, we must regard that the following four conditions are satisfied at each step, in order to have a convergent method:
for $j = 0,..,N$
for $i = 2,..,m$
$r_{l,j} - \beta_{l,i,j} > 0$, $\quad 1 - 2r_{l,j} + \beta_{l,i,j} > 0$,

for $i = m+2,..,2m$

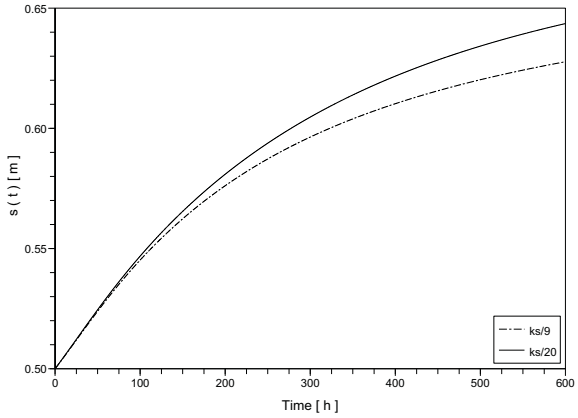

Figure 5: Free boundary position ($t \leq 600h$)

$r_{s,j} - \beta_{s,i,j} > 0$, $\quad 1 - 2r_{s,j} + \beta_{s,i,j} > 0$. The numerical solutions obtained from the above algorithm, are used to find counterexamples that show the non-monotonicity of the solution with respect to the solid thermal conductivity. For this reason, any analysis about stability or convergence are done.

B. Numerical results

In order to analyze the monotony, we consider the numerical solutions of problem (Pq) $(q = 2,3)$, for the thermal coefficients of water (Alexiades and Solomon, 1996):
$*$ mass density $\rho = 1000\ \frac{kg}{m^3}$;
$*$ latent heat $\ell = 334000\ \frac{J}{kg}$;
$*$ thermal conductivity of liquid phase $k_l = 0.58\ \frac{W}{mK}$;
$*$ thermal conductivity of solid phase $k_s = 2.24\ \frac{W}{mK}$;
$*$ specific heat of the liquid phase $c_l = 4185.5\ \frac{J}{kgC}$;
$*$ specific heat of the solid phase $c_s = 2090\ \frac{J}{kgC}$.

i) Problem $(P2)$

We consider the numerical solutions of problem $(P2)$ for two different values of the thermal conductivity of the solid phase: $\frac{1}{20}k_s$, $\frac{1}{9}k_s$. For the initial conditions and the boundary conditions we take:
$\varphi(x) = -76x + 38$; $\quad \psi(x) = -8x + 4$
$f(t) = -10$; $\quad g(t) = -8$;
and we choose $\triangle t = 240seg$ and $m = 25$.

In Figure 5, we obtain a comparative graphic of the free boundary positions and in Figure 6 we show the temperature distribution at a fixed time $t = 600hs$.

We can observe that the free boundary position have a monotone behavior but it does not occur for the temperature distributions.

For the numerical solutions of the problem $(P3)$ we consider the same values of the thermal conductivity of the solid phase as above. For the initial conditions and the boundary conditions we take:
$\varphi(x) = -140x + 70$; $\quad \psi(x) = -8x + 4$

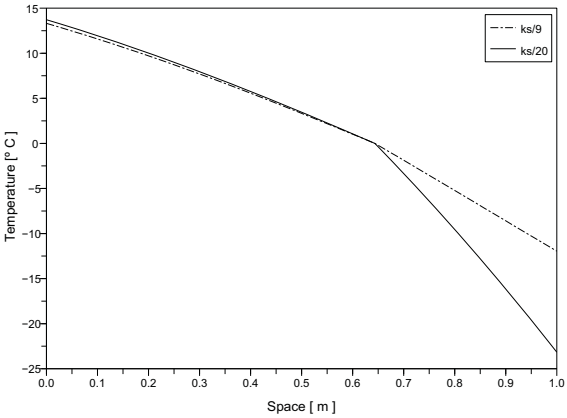

Figure 6: Temperature distribution at $t = 600h$

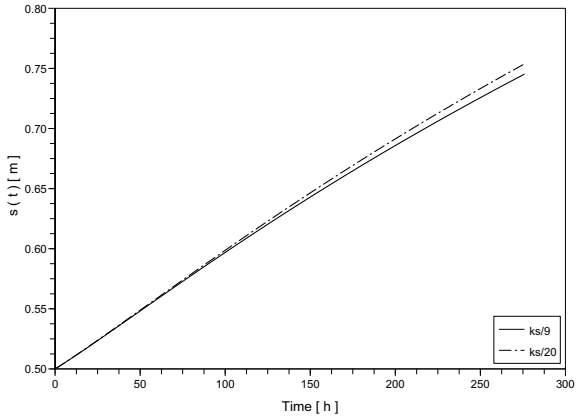

Figure 7: Free boundary position ($t \leq 276h$)

ii) Problem ($P3$)

$f(t) = 70;$ $\quad g(t) = -8;$

and we choose $\triangle t = 240seg.$ and $m = 25$.

In Fig. 7, we obtain a comparative graphic of the free boundary positions, in Figure 8 we show the temperature distribution at a fixed time $t = 276hs$ and in Figure 9 we zoom in Figure 8. We can observe that the free boundary position have a monotone behavior but it does not occur for the temperature distributions.

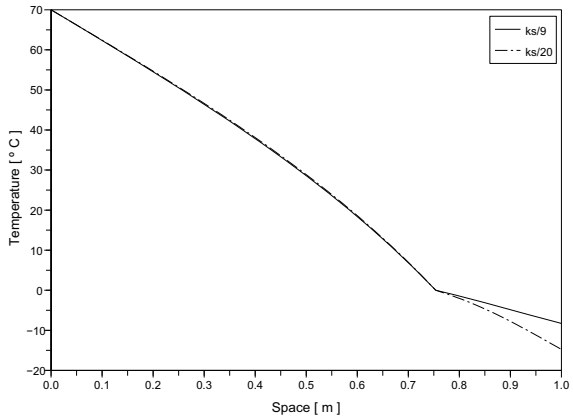

Figure 8: Temperature distribution at t= 276 h

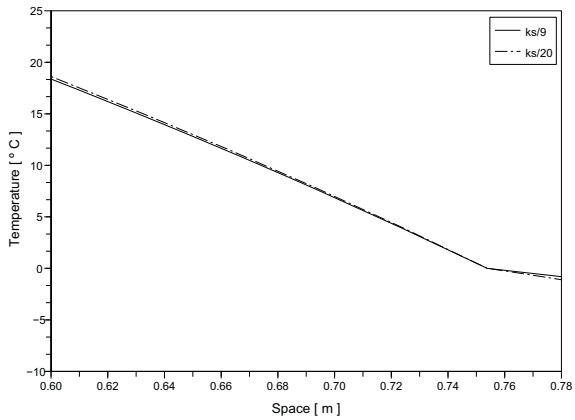

Figure 9: Temperature distribution at t= 276 h

VI. CONCLUSIONS

It was showed, through the maximum principle, that the solutions of two-phase one-dimensional Stefan problems for finite domains, depend monotonically on the latent heat, the mass density, the specific heat of each phase and the thermal conductivity of liquid phase when we consider the following boundary conditions:

1) temperature specification on both fixed faces;

2) flux specification on both fixed faces;

3) temperature specification on the left face and flux specification on the right face.

We developed a numerical scheme using finite difference methods with variable space step in order to obtain an approximate solution for those problems. We have showed, through those approximate solutions, that there is no monotony of the solution when the thermal conductivity of the solid phase changes for the problems with boundary conditions 2) and 3).

The monotony of the solution for the problem with temperature boundary condition on both fixed faces when the thermal conductivity of the solid phase changes, is at the moment, an open problem.

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