When Does a Hyperbola Meet Its Asymptote?: Bounded Infinities, Fictions, and Contradictions in Leibniz

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Abstract: In his 1676 text *De Quadratura Arithmetica*, Leibniz distinguished *infinita terminata* from *infinita interminata*. The text also deals with the notion, originating with Desargues, of the point of intersection at infinite distance for parallel lines. We examine contrasting interpretations of these notions in the context of Leibniz's analysis of asymptotes for logarithmic curves and hyperbolas. We point out difficulties that arise due to conflating these notions of infinity. As noted by Rodríguez Hurtado *et al.*, a significant di-

fference exists between the Cartesian model of magnitudes and Leibniz's search for a qualitative model for studying perspective, including ideal points at infinity. We show how respecting the distinction between these notions enables a consistent interpretation thereof.

Key–words: infinitesimal calculus, useful fiction, infinity, infinitesimals, ideal perspective point.

¿Cuándo una hipérbola encuentra a su asíntota? Infinitos acotados, ficciones y contradicciones en Leibniz

Resumen: En su texto *De Quadratura Arithmetica*, de 1676, Leibniz distinguió *infinita terminata* de *infinita interminata*. Asimismo, el texto se ocupa de la noción, que se origina con Desargues, del punto de intersección a una distancia infinita para las rectas paralelas. En este trabajo, examinamos interpretaciones enfrentadas de estas nociones en el contexto del análisis que hace Leibniz de las asíntotas para hipérbolas y curvas logarítmicas. Señalamos las dificultades que surgen de combinar estas nociones de infinito. De acuerdo con lo que observan Rodríguez Hurtado *et al.*, hay una diferencia significativa entre el modelo cartesiano de magnitudes y la búsqueda de Leibniz de un modelo cualitativo para estudiar la perspectiva, incluyendo puntos ideales en el infinito. Finalmente, mostramos cómo respetar la distinción entre estas nociones permite una interpretación consistente de las mismas.

Palabras clave: cálculo infinitesimal, ficción útil, infinito, infinitesimales, punto de perspectiva ideal.

1. Bounded and unbounded infinity

A key distinction in Leibniz's approach to geometry and the calculus is that between bounded infinities (*infinita terminata*) and unbounded infinities (*infinita interminata*). As noted by Knobloch, the distinction was elaborated in Proposition 11 of his treatise *De Quadratura Arithmetica* (*DQA*):

> [Leibniz] distinguished between two infinites, the bounded infinite straight line, the *recta infinita terminata*, and the unbounded infinite straight line, the *recta infinita interminata*. He investigated this distinction in several studies from the year 1676. Only the first kind of straight lines can be used in mathe-

matics, as he underlined in his proof of theorem 11 [i.e., *Propositio* XI] (Knobloch 1999: 97).

Leibniz mentioned the distinction in a 29 July 1698 letter to Bernoulli. Here Leibniz analyzes a geometric problem involving unbounded infinite areas and states an apparent paradox: "Therefore the two infinite spaces are equal, and the part is equal to the whole, which is impossible" (Leibniz 1698: 523. Translation ours).

To resolve the paradox (i.e., the clash with the part–whole principle), Leibniz exploits the notion of bounded infinity:

> Properly speaking, the last (ultima) abscissa $A\ _0B$ is not null, as if O were falling on A, and the last (ultima) ordinate $_0B\ _0C$ is not unbounded (*interminata*), as if $_0B\ _0C$ were falling on the asymptote. Rather, $A\ _0B$ is infinitely small, and $_0B\ _0C$ is infinitely large, but bounded (*terminata*) (Leibniz 1698: 523)[1].

As noted by Knobloch (1994: 267–268), Leibniz also mentioned the distinction in his February 1702 correspondence with Varignon (Leibniz 1702: 91). The distinction enabled him to avoid contradicting the part–whole principle while still employing infinite magnitudes in analysis and geometry.

1.1. Inconsistency of Maxima and Minima

Leibniz used the term *Maxima* to refer to infinite wholes, and the term *Minima* to refer to points viewed as constituent parts of the continuum. Already in his 1672/3 text "On Minimum and Maximum," Leibniz rejected both Minima and Maxima in the following terms:

> *Scholium.* We therefore hold that two things are excluded from the realm of intelligibles: minimum and maximum; the indivisible, or what is entirely one, and everything; what lacks parts, and what cannot be part of another (Leibniz 2001: 13. Tr. R Arthur).

Leibniz's rejection of Maxima amounts to the rejection of infinite wholes (e.g., unbounded lines) as inconsistent, while the rejection of their

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[1] Note that the subscripts are on the left in $_0B$ and $_0C$, as elsewhere in the sequel.

counterparts, Minima, amounts to the rejection of putative simplest consti-
tuents of the continuum, i.e., the rejection of a punctiform continuum. To
Leibniz, points play only the role of endpoints of line segments. Thus we
find in the Scholium to Proposition 11: "The magnitude of an unbounded
line, just as that of a point, is beyond the realm of geometric considerations"
(Leibniz 2012: 549. Translation ours).

An example of an unbounded infinity is an unending line, or the con-
tinuum. Such infinities, when taken as a whole, were considered by Leibniz
to lead to a contradiction with the part-whole principle, and therefore of
little use in geometry and calculus.

Unlike unbounded infinity, bounded infinity has a pair of endpoints.
With reference to a line, a Leibnizian bounded infinity can be thought of as
a segment with infinitely separated endpoints[2].

In the Scholium to Proposition 11, Leibniz speaks of "linea terminata
quidem, infinita tamen" (Leibniz 2012: 549) i.e., a bounded infinite line.

1.2. Proposition 11

In Proposition 11 of *DQA*, Leibniz discusses bounded infinities and
explains their reciprocal relation to infinitesimals. He uses the notation $(\mu)\mu$
for an infinitesimal. Here μ can be thought of as the origin[3]. Leibniz notes
that in order to show that a figure of infinite length bounded by a curve λ
may have a finite magnitude one must proceed as follows:

> Substitute for a line $\mu\lambda$ [the asymptote] a line $(\mu)\lambda$, the point (μ) being taken
> just above μ and the interval $(\mu)\mu$ being infinitely small, so that the ordinate
> $(\mu)\lambda$ will be of infinite length (Leibniz 2012: 547. Translation ours).

Here the bounded infinity, denoted $(\mu)\lambda$, is the ordinate of the point
on the curve λ corresponding to an infinitesimal abscissa (μ). Leibniz goes
on to emphasize that "$(\mu)\lambda$ will not be the asymptote" (Leibniz 2012: 547.
Translation ours). Thus bounded infinity is distinct from the asymptote.
Note that the line $(\mu)\lambda$ is a subline of the infinite unbounded line with
abscissa (μ).

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[2] See section 3, item 3.
[3] See figure on next page.

(μ) $(\mu)\lambda$

μ **asymptote** $\mu\lambda$
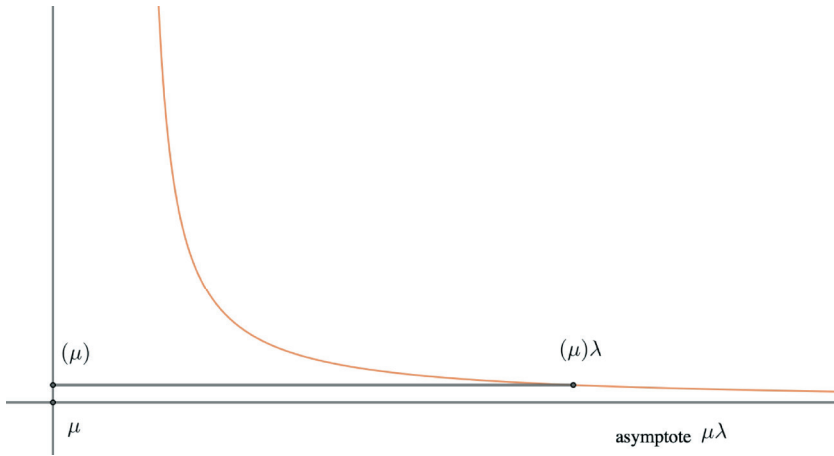
1.3. Scholium

In the Scholium to Proposition 11, Leibniz points out a further difference between a bounded infinity and an unbounded infinity:

> Therefore one cannot say that a bounded line is a geometric mean between a point, which is the Minimum line, and an unbounded line, which is the Maximum line. But one can say that a finite line is the geometric mean, in a sense not approximate but precise, between an infinitely small line and an infinite line [i.e., bounded infinity] (Leibniz 2012: 549. Translation ours).

Thus, the geometric mean of an infinitesimal and a bounded infinity can turn out to be a finite quantity. Infinitesimals and bounded infinities satisfy the usual rules of arithmetic, which is not the case for unbounded infinity.

Leibniz holds that (unlike unbounded infinity) bounded infinity is useful in geometry and calculus, and repeatedly describes it as a fiction. Both the nature of bounded infinity and the exact meaning of its fictionality are subject to current debate among Leibniz scholars[4].

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[4] See, e.g., Rabouin and Arthur (2020), Eklund (2020), and Esquisabel and Raffo Quintana (2021).

2. Intersection point interpreted

Several scholars have commented on Leibnizian points at infinity, including Knobloch, Rabouin, and Arthur. We will examine Knobloch's position in Section 2.1, and that of Rabouin and Arthur, in Section 2.3.

2.1. Hyperbola and its asymptote

Knobloch comments on Leibniz's analysis in *DQA* of a hyperbola c and its asymptote d in 1990 as follows:

> For each hyperbola of any degree, both of the axes are asymptotes. They don't meet, or rather they don't meet until after an infinitely long interval, with the curve. Therefore one must conclude: d is an asymptote − either d ∩ c = ∅ (theorem 11, scholium: never; theorem 23: nowhere), or alternatively d ∩ c ≠ ∅ (theorem 45) (Knobloch 1990: 48. Translation ours).

Knobloch goes on to describe the first case (d ∩ c = ∅) as occurring for unbounded infinity, and the second case (d ∩ c ≠ ∅), for bounded infinity:

> Nevertheless, this is not a contradiction because, according to the Leibnizian explanation, infinity can be unbounded or bounded. If one has in mind the first possibility, one obtains the perfect asymptote which does not meet the curve . . . If one has in mind the second possibility, one obtains a point common to the line and the curve, after an infinitely long interval. Leibniz exploits both conceptions (Knobloch 1990: 48. Translation ours).

Thus, Knobloch claims that the hyperbola and its asymptote have nonempty intersection when the asymptote is a bounded infinity. He bases his claim upon *DQA*, theorem 45 (mentioned in his second case). The relevant passage from theorem 45 is analyzed in Section 3 below. This is the unique piece of evidence presented by Knobloch in favor of his hypothesis of identification of bounded infinity and perspective point at infinity in Leibniz.

Such a claim of nonempty intersection is repeated in 1994 (Knobloch 1994: 266). Here Knobloch writes, reasonably enough, that

> In order to avoid contradictions we have to try to understand the explanations of an author in their contexts. I would like to try to demonstrate that Leibniz

dealt with the infinite in a consistent manner, although the contrary seems to be the case. Let us consider three examples, etc. (Knobloch 1994: 265).

Of Knobloch's three examples of an apparent contradiction, the first is identical to his pair "d∩c = ∅, d∩c ≠ ∅" analyzed in his 1990 article on page 48. Knobloch concludes that bounded infinity is a fictional entity: "A bounded infinite quantity is a fictitious quantity on which we rely if we measure infinitely long but finite spaces" (Knobloch 1994: 267).

One finds a similar comment in 1999: "[Leibniz] assumed a fictive boundary point on a straight halfline which is infinitely distant from the beginning: a bounded infinite straight line is a fictitious quantity" (Knobloch 1999: 97).

2.2. Reliability and recourse to fictionalism

However, it is unclear how describing the *infinita terminata* as fictions could explain their reliability in mathematical reasoning. Jesseph voices a similar concern in the following terms: "[T]he recourse to fictionalism is insufficient on its own to make a demonstration employing such fictions truly rigorous or convincing" (Jesseph 2015: 196).

From the viewpoint of Knobloch's interpretation, it is difficult to develop a coherent reading of the primary sources in Leibniz with regard to bounded infinities and perspective points at infinity.

2.3. Infinite quantities and ideal points

Similarly to Knobloch, Rabouin and Arthur blend infinite quantities of the Leibnizian infinitesimal calculus and the perspective points at infinity à la Desargues and Pascal. They write:

> [E]ven a "clear and distinct" concept—i.e. one for which we can provide a nominal definition (allowing us to distinguish the entity in question) may harbour a hidden contradiction, which appears when analysing all of its constituents. Still, one can use such concepts for deriving truth. This is the case with the notion of a mathematical fiction applied to an infinite quantity (Rabouin and Arthur 2020: 406).

They elaborate on infinity in relation to the point at infinity of Desargues and Pascal as follows:

The parallel with a point at infinity may be recalled here since this is a notion which produces a contradiction when inserted in some proofs of Euclid's Elements (such as I, 27, where we assume that parallel lines meet), but which is also useful (when accompanied with suitable demonstrations) in order to produce general geometrical truths, such as the ones promoted by Desargues and Pascal (Rabouin and Arthur 2020: 407, note 14).

Rabouin and Arthur go on to provide the following analysis of Leibnizian infinitesimals:

Thus, when Leibniz says that he understands the infinitely small to be a fiction, this is not a way of deflecting criticism by simply abjuring infinitesimals, as is sometimes assumed. It means that even though its concept may contain a contradiction, it can nevertheless be used to discover truths, provided a demonstration can (in principle) be given to show that its being used according to some definite rules will avoid contradiction (Rabouin and Arthur 2020: 407).

They reiterate the claimed connection between the infinitely small and the projective point at infinity:

(1) "All of this is crucial for a proper reading of Prop. 8, the one in which the fiction of 'infinitely small' entities will be used for the first time" (Rabouin and Arthur 2020: 418).

(2) "The idea of fiction is mentioned a first time for designating the 'point at infinity' introduced by geometers developing projective considerations, such as Desargues and Pascal (schol. VII)" (Rabouin and Arthur 2020: 418, note 42).

In fact the *infinite parvae* are briefly mentioned in Proposition 8 but are not used. Instead, Leibniz gives an exhaustion argument, and concludes in the Scholium: "I went into all these details to allow geometers who encounter a similar reasoning to avoid engaging with it, without however running the least risk" (Leibniz 2012: 549. Translation ours). Leibniz does not introduce the notation $(\mu)\mu$ for an infinitely small line until Proposition 11.

The alleged connection between infinitesimals and Desargues is mentioned yet again: "The parallel between the introduction of point at infinite distance and infinitesimals […] appears in the *DQA* and was already present in Desargues […]" (Rabouin and Arthur 2020: 421, note 52).

A debate of long standing concerns the issue of the fictionality of the Leibnizian infinitesimal and infinite quantities. In his 1990 and 1994 articles, Knobloch seeks to relate such fictionality to alleged paradoxical behavior of bounded infinity when the hyperbola and its asymptote are said to meet,

while Rabouin and Arthur read the fictionality of infinitesimals as incon-sistency in 2020. In Sections 3 and 5 we will analyze the interpretation of fictionality and of the intersection point between a curve and its asymptote.

3. *Theorem 45 of* **De Quadratura Arithmetica**

Knobloch's interpretation relies on a reading of a sentence in the proof of theorem 45 (i.e., Propositio XLV) of *DQA*. The sen-tence reads as follows: "It remains to show that the line Cβ etc. represents an asymptote, i.e., that it cannot meet the logarithmic curve ARST etc. except at an infinite distance" (Leibniz 2012: 633. Translation ours).

The issue is the nature of the intersection between the asymptote denoted "Cβ etc." and the logarithmic curve denoted "ARST etc." Leibniz supplements the notation for a line by the abbreviation "etc." to indicate that he is referring to an unbounded line rather than a bounded infinity. Thus, he uses the notation "Cβ etc." three times in the proof of theorem 45 to denote the unbounded asymptote. The figure illustrating the theorem[5] marks the finite points C and β lying on the asymptote to the curve. See figure below.
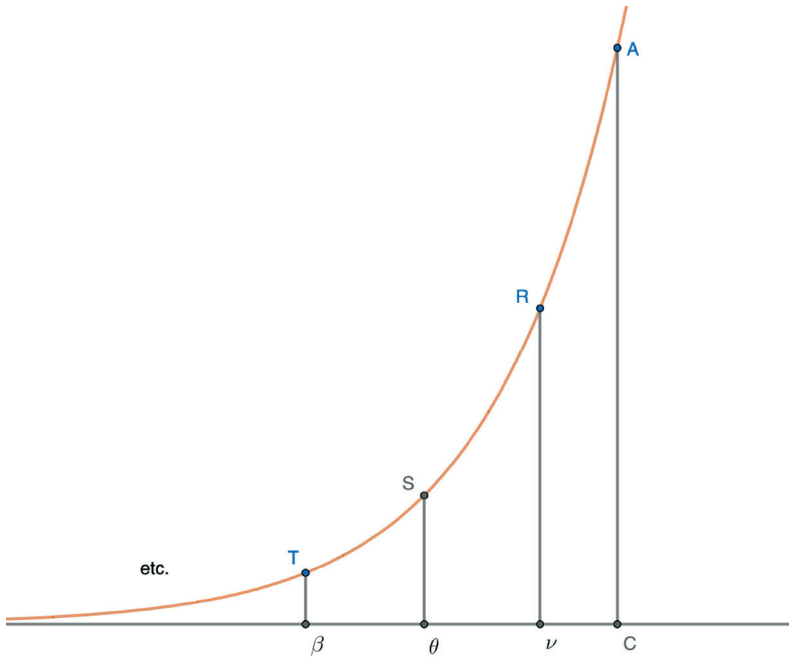
[5] I.e., Figure 14 (Leibniz 2012: 624).

We note the following points:

• The term bounded infinity is not used in the proof of theorem 45.

• The intersection at infinity between the logarithmic curve denoted ARST etc. and its asymptote is only mentioned in passing in a double negation: (A similar double negation referring to the intersection at infinity of a hyperbola and its asymptote occurs in the proof of Proposition 22): "can not meet… except at an infinite distance".

Knobloch's paraphrase of the phrase transforms the double negative clause into a positive one: "Second assertion: The straight line Cβ etc. is an asymptote, which is (*seu*), it can meet (*occurrere posse*) the logarithmic curve ARST only after an infinite interval (*infinito abhinc intervallo*)" (Knobloch 2018b: 28).

We can therefore make the following five remarks.

1. If one wishes to explain why, according to Leibniz, bounded infinity is useful in geometry while unbounded infinity is not, theorem 45, for all its intrinsic interest, is of little help.

2. In Proposition 11, both the unbounded asymptote and a genuine bounded infinite (μ)λ occur in the same proof, and Leibniz repeatedly states that they are distinct.

3. Using the terminology of Proposition 11, Knobloch acknowledges the existence of an infinitely thin rectangle formed by drawing a pair of lines, parallel to the axes, and passing through a point infinitely far along the curve λ with infinitesimal abscissa:

> If the abscissa μ(μ) is infinitely small, the ordinate (μ)λ is infinitely long, namely greater than any line that can be specified (*designable*) ; the rectangle formed by the infinite line and the infinitesimal line equals to a finite, constant square according to the nature of the hyperbola (Knobloch 1990: 41. Translation ours).

The side (μ)λ is an instance of a bounded infinity. It is a subline of the unbounded infinite line which passes through the infinitesimal abscissa (μ) and is parallel to the asymptote. By the nature of such a rectangle (infinitely thin, infinitely long), there is always a nonzero distance from the vertex (of the rectangle given by a point on the curve) to the asymptote, even when the point is infinitely close to the asymptote because (μ) is infinitesimal.

4. Leibniz observes that bounded lines (i.e., segments) cannot exhaust the unbounded infinite line. (The full observation reads as follows:

> Indeed, just as one does not change a bounded line by adding to or removing from it some points, even infinitely many, so also by reproducing

any number of times a bounded line, one can neither constitute nor exhaust an unbounded line. It is otherwise for a bounded infinite line, which can be conceived as formed of a multitude of finite lines, even though such a multitude exceeds all number (Leibniz 2012: 549. Translation ours).

The claim that they cannot exhaust it presupposes in particular that it is meaningful to envision an attempt to carry out such an exhaustion; namely that a bounded line, while unable to exhaust it, is a subline of the unbounded infinite line (as in the example (μ)λ mentioned in item 3 above).

If so, how could the unbounded infinite line Cβ etc. have empty intersection with the logarithmic curve ARST etc. (as claimed by Knobloch; see Section 2) whereas its subline nonetheless manages to meet this curve?

5. Leibniz's *infinita terminata* can be naturally scaled; they can be multiplied by 2, 3, … just as the infinitesimal (μ) can be divided by 2, 3, …, and satisfy the usual rules of arithmetic (see Section 1.3). One cannot obtain such properties by adjoining a single point at infinity, as in projective geometry, implied in Leibniz's comment quoted in Section 2. It is unclear how calling it "fictional" could help here.

It is therefore difficult to develop a consistent reading of the Leibnizian notions of bounded infinity and perspective point at infinity from the viewpoint of Knobloch's interpretation that seeks to identify them.

Projective geometry was only in its incipient stages at the time, but Leibniz did speak of ideal meeting points at infinity for parallel lines (see Section 5). We argue that, if the hyperbola and its asymptote meet at infinity, it is only in the sense familiar from projective geometry, a sense distinct from the *infinita terminata* of Leibniz's geometry and calculus. We review the history of the idea of the perspective point at infinity in Section 4.

4. Infinite distance from Kepler to Desargues

By the last quarter of the 17th century when Leibniz started work on *DQA*, the idea of an infinitely distant point in the geometry of perspective was already a familiar one, due to the work of Kepler, Desargues, and Bosses.

In 1604, Kepler referred to an infinitely distant point associated with a pencil of parallel lines as a "blind focus." He held that

In the Parabola one focus, D, is inside the conic section, the other is to be imagined either inside or outside, lying on the axis [of the curve] at an infinite distance from the former (*alter vel extra vel intra sectionem in axe fingendus*

est infinito intervallo à *priore remotus*), so that if we draw the straight line HG or IG from this blind focus (*ex illo caeco foco*) to any point G on the conic section, the line will be parallel to the axis DK[6].

According to Debuiche, the idea that the projective closure of a line is a circle can already be detected in Desargues (1639. Reprint: Taton 1951):

> Desargues presents the idea of a complete correspondence between a straight line and a circle, since a straight line can be considered as a circle closed in on itself at an infinite distance (Debuiche 2013: 373).

Field and Gray note: "Desargues began with the remark that lines will be supposed to contain a point at infinity, which may be reached by travelling in either direction along the line" (Field and Gray 1987: 47).

In connection with Leibniz's work on perspective, Rodríguez Hurtado *et al.* note that "The characterisation of the point of view as the meeting point in the infinity of the parallels comes from Arguesian perspective.[56]"[7].

The reference is to Desargues' 1639 *Brouillon Project*. Based on Leibniz's 1679 letter to Huygens, Rodríguez Hurtado *et al.* point out a significant difference between the Cartesian model of magnitudes and Leibniz's search for a qualitative model for studying perspective:

> In contrast to the Cartesian algebraic model, centred on the determination of magnitudes (analysis of quantitative relations), Leibniz wanted to construct a qualitative model, based on analysis of position (*situm*) (Rodriguez *et al.* 2021: 4).

While critical of certain aspects of the article by Rodríguez Hurtado *et al.*, a recent text by Debuiche and Brancato confirms that "perspective science can be understood as containing the 'whole *Geometria Situs*' since the geometry of situation only deals with the mutual positions of points in space" (Debuiche and Brancato 2023: 67). Accordingly, perspective points at

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[6] *Ad Vitellionem Paralipomena*, cap. IV n. 4: *De coni sectionibus*. See further in Field (1986: 450), Field and Gray (1987: 186-187) and Del Centina (2016: 568).
[7] Rodríguez Hurtado *et al.* (2021: 16). Their note 56 reads: "56 It is worth mentioning that Leibniz transcribes the definition of point of view that Desargues makes at the end of the *Brouillon Project* (included in A VII 7: 111)". The page number given is incorrect. It should be A VII 7, item 65, page 593.

infinity are not expected to have the properties of magnitudes, unlike Leibnizian *infinita terminata*. Leibniz explicitly acknowledged a debt to Desargues in a 1692 publication in *Act. Erudit. Lips.* entitled *De Linea ex Lineis Numero Infinitis*:

> Geometers customarily refer as ordinates to parallel lines in any number, traced between a curve and a fixed line (*directrix*); when they are perpendicular to the latter (which then plays the role of an axis), one refers to ordinates par excellence. Desargues generalized this by considering also as ordinates, the lines which converge toward a unique common point or diverge from it. Parallel lines can be considered converging or diverging lines, if one fictively considers that their common point is at an infinite distance (Gerhardt 1850-63, V: 266-267. Translation ours).

Leibniz deepened his knowledge of projective methods in Hannover by studying Abraham Bosse's writings on Desargues as noted in (Rodriguez *et al*, 2021), but the more detailed statement in 1692 is merely a clearer elaboration of the position already found in *DQA*.

In Section 5, we will analyze the occurrence in Leibniz of the Arguesian point at infinity to which parallel lines 'diverge'.

5. Leibniz's pencil of parallel lines

In the Scholium to Proposition 7 of *DQA*, Leibniz speaks of a pencil of parallel lines meeting at a fictional point A. We argue that this is a fiction in a sense distinct from his infinitesimals and *infinita terminata*. Leibniz attributes the idea to "illustrious geometers":

> Furthermore, illustrious geometers having undertaken to study the conics from a general viewpoint, call ordinates to curves not only, as is common, the parallel lines $_1C$ $_1B$, $_2C$ $_2B$, $_3C$ $_3B$, but also the lines A $_1C$, A $_2C$, A $_3C$ which all converge toward a unique point A (which is entirely correct since one can, without committing an error, consider parallel lines as convergent lines, up to considering fictively (*fingatur*) that their point of intersection or their common center is at an infinite distance, as in the case of the focus or vertex of the parabola (Leibniz 2012: 538. Translation ours).

Leibniz's reference to a parabola can be interpreted as follows. If one sends rays out of the (finite) focal point of the parabola, then after bouncing off the parabola, the rays turn into a pencil of parallel lines.

The corresponding ideal point at infinity A is the "focal point at infinity" of the parabola. See Section 4 for a discussion of possible antecedents in Kepler and Desargues.

The illustrious geometers are identified as Pascal and Desargues by the editor of Leibniz (2004: 73).

A curve (such as a hyperbola or a logarithmic curve) and its asymptote are obviously not a pair of parallel lines, but they meet at infinity at a unique fictional point A determined by the pencil of lines parallel to the asymptote.

To illustrate that the perspective point at infinity can be easily formalized as the ideal point of intersection between the logarithmic curve and its asymptote, we outline a modern formalisation in the case of the hyperbola $xy = 1$ and its horizontal asymptote $y = 0$ in the affine plane. Passing to homogeneous coordinates $[x_1, x_2, x_3]$ where $x = x_1/x_3$ and $y = x_2/x_3$, we obtain the equation

$$x_1 x_2 = (x_3)^2$$

for the projective completion of the hyperbola, and equation $x_2 = 0$ for that of the asymptote. The point at infinity for the asymptote is the point A = [1, 0, 0]. The point A clearly lies on the curve $x_1 x_2 = (x_3)^2$ of the equation displayed above, as well, and can therefore be thought of as the ideal point of intersection at infinity between the hyperbola and its horizontal asymptote.

What could be the relation between such an ideal point at infinity A and Leibniz's bounded infinities? We make the following two observations.

(1) Bounded infinity $(\mu)\lambda$, as well as the infinitesimal $(\mu)\mu$, are only discussed by Leibniz in Proposition 11. Therefore, the mention of the ideal point A earlier in the text, namely in the Scholium following Proposition 7, could not be related to bounded infinity, unless we presume Leibniz to be sloppy in his exposition in *DQA* by using a concept before discussing it.

(2) The ideal point at infinity A in projective geometry does not make sense as an infinite magnitude, because it cannot occur as an element in an ordered system that is greater than all the other magnitudes. Indeed, such a point A is assigned to a pencil of unoriented (undirected) lines; if A were greater than all the other magnitudes, it would also have to be declared smaller than all the other magnitudes, leading to an absurdity.

Thus if one starts with an affine line R, the corresponding projective line RP^1 is the circle S^1 which admits no natural structure of an order. See Section 4 for a discussion of possible antecedents in Desargues.

Anglade and Briend note that

Desargues formulates […] an analogy between the the circle and the line, which suggests that he had an accurate image of what one refers to today as the (real) projective line as being topologically a circle (Anglade and Briend 2017: 550. Translation ours).

Anglade and Briend published a series of in-depth studies of the *Brouillon Project* culminating in their 2022 work (Anglade and Briend 2022).

Thus, Leibniz's (projective) ideal point at infinity A cannot be identified with an *infinitum terminatum* without creating unnecessary inconsistencies. An *infinitum terminatum*, being the inverse of an infinitesimal, cannot be a perspective ideal point at infinity.

There are hints in Knobloch's later work that he is aware of the difficulties with his interpretation of Leibnizian infinitesimals, as when he writes:

The error is smaller than any assignable error and therefore zero […]. Such an error necessarily is equal to zero as Leibniz rightly states. For if we assume that such an error is unequal to zero it would have a certain value. But this implies a contradiction against the postulate that the error has to be smaller than any assignable quantity, that is, also smaller than this certain value. Yet, Leibniz explicitly calls such errors infinitely small: *We should not try to make things seem better* (Knobloch 2018a: 12).

Leibnizian infinitesimals were not assignable, as made clear by Leibniz himself, who wrote: "Even though they are not assignable, they turn out to be something existing and not an absolute nothing" (Leibniz as quoted in Bella 2019: 195. Translation ours).

6. Conclusion

We have sought to redress a conflation of Leibniz's notion of bounded infinity and the notion of the perspective (projective) point at infinity, in the recent literature on Leibniz. We argued that Leibniz's proof of theorem 45 of his *De Quadratura Arithmetica* provides no evidence for relating his notion of bounded infinity to a remote point of intersection (ideal projective point) of a curve (hyperbola or logarithmic curve) and its asymptote.

Furthermore, the hypothesis of such a point of intersection at bounded infinite range is mathematically incoherent. Postulating that there are only two types of infinity in Leibniz – bounded and unbounded – leads to a paradoxical conclusion that what is identifiably the projective ideal

point at infinity of a pencil of parallel lines, must be bounded infinity. Such an approach leads to unnecessary inconsistencies (as detailed in Section 3 and 5). Conflating perspective points at infinity and *infinita terminata* amounts to what Leibniz may have called a "category error": the former belong in analysis situs whereas the latter belong in analysis of magnitudes.

As noted by Jesseph (see Section 2.2), appeals to fictionality are insuf‐ficient to provide an adequate account of Leibniz's mathematical procedures. More convincing accounts of the fictionality of infinitesimals were recently developed e.g., by Eklund (2020), as well as Esquisabel and Raffo Quintana (2021). Interpreting Leibniz's infinitesimals is an area of lively debate. In 2021, Bair *et al.* published a comparative study of three interpretations (Bair *et al.* 2021). Katz *et al.* presented three case studies in Leibniz scholarship (Katz *et al.* 2021). In 2022, Katz *et al.* presented and analyzed a pair of rival approaches (Katz *et al.* 2022). In the same year, Archibald *et al.* formulated some criticisms (Archibald *et al.* 2022). In 2023, Bair *et al.* (2023) provided both a brief response and a detailed response (2022). A detailed study of Leibnizian methodology appeared in (Katz *et al.* 2023).

In sum, one is led to recognize that there are multiple types of infinity in Leibniz's geometry and calculus. We conclude that an ideal point at in‐finity (associated with a pencil of parallel lines) is borrowed from Desargues, and is a type distinct from the *infinita terminata*. Such an approach enables a consistent interpretation of the Leibnizian notions of perspective point at infinity and bounded infinity, and explains how the latter can satisfy the usual rules of arithmetic such as scalability and invertibility.

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