REVISTA DE LA
UNIÓN MATEMÁTICA ARGENTINA
Volumen 46, Número 2, 2005, Páginas 45–72

EXTENDING POLYNOMIALS ON BANACH SPACES—A SURVEY

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Introduction

Throughout, E, G, and F will be complex Banach spaces, E a subspace of G. The question which gave rise to the material reviewed here was first posed by Dineen [D1] in relation to holomorphic completeness and in the context of more general locally convex spaces. It is the following: when can a continuous homogeneous polynomial $P : E \longrightarrow F$ be extended to $\overline{P} : G \longrightarrow F$? Several answers—none complete—have been given for varying hypotheses on the spaces and the polynomials involved, and several applications have appeared. We attempt here to unify the existing approaches to the problem, and to point out the common ingredients in the solutions.

It is clearly not possible in general to extend polynomials to larger spaces, in fact this is true even of linear functions: the identity $c_0 \longrightarrow c_0$ cannot be extended to $\ell^\infty \longrightarrow c_0$, for c_0 is non-complemented in ℓ^∞. When F is the complex field and P is linear there is of course the Hahn-Banach theorem, but we shall see that in the non-linear case, there are scalar-valued polynomials which cannot be extended.

The positive results in this matter can be loosely grouped into two main categories: those providing linear extension morphisms by which all polynomials can be extended to some larger space, and Hahn-Banach type extensions applicable to some polynomials and extending those to all larger spaces. We concentrate on the first type of result in sections 1 and 2, and on the second type in section 3.

We restrict our attention to polynomials over Banach spaces, although many results have been obtained regarding the extension of holomorphic functions on Banach or other locally convex spaces ([Bol], [CM], [GGM2], [Kh1], [Kh2], [Mo2], [Mo3], [MV]). Also, many applications have appeared which we do not go into here: to holomorphic completeness [D1], to geodesics [DT], polarization constants [LR], reflexivity of spaces of polynomials [JPZ] [AD], the spectra of algebras of analytic functions over Banach spaces [ACG1] [AGGM], integral representation of analytic functions [PZ], and orthogonally additive polynomials ([BLLl], [PgV], [CLZ]).

In the first section we consider the Aron-Berner extension. We begin with the Arens product in a commutative Banach algebra, a very specific extension of a bilinear function, but nevertheless an extension in which some of the main points of more general extensions already appear, such as lack of symmetry and the notion of regularity. We then define and study the Aron-Berner extension, an extension of polynomials from a Banach space to its bidual. In the second section, we consider extensions from E to G. Here all solutions stem from the existence of a continuous linear extension morphism for linear forms $E' \longrightarrow G'$, a condition obviously stronger than Hahn-Banach, and not satisfied in all cases. Section 3 is devoted to Hahn-Banach type extensions. We are naturally drawn to 'linearization' of polynomials, and thus to preduals of spaces of polynomials. The space of 'extendible' polynomials is considered also.

This survey has been brewing, on and off, for about a decade. It started out as a talk prepared for the Conference on Polynomials and Holomorphic Functions on Locally Convex Spaces, in University College Dublin in September of 1994, and then lay dormant for a couple of years until the Departamento de Análisis Matemático of the Universidad Complutense de Madrid invited the author to lecture on these topics in the winter of 1997, thus providing the oportunity for extensive rewriting. A first version appeared as a preprint [Z3] in the series *Publicaciones del Departamento de Análisis Matemático* of the Complutense. More recently, the author has lectured on these matters in a Seminar at the Universidad de Buenos Aires, and so had the oportunity to add more recent material —particularly to section 3— and make these notes up to date. It is a pleasure to thank my colleagues at UCD, UCM, and UBA for their interest and support. Also, many participants at the Conference in Dublin, and many of those present at the Madrid and Buenos Aires lectures have helped to mold these notes with their comments, their questions, and in more than a few cases, their answers. In this respect, it is a pleasure to acknowledge the contribution of J. M. Ansemil, R. Aron, F. Bombal, C. Boyd, D. Carando, V. Dimant, S. Dineen, J. A. Jaramillo, J. G. Llavona, A. Peris, A. Prieto, R. Ryan, S. Lassalle, and A. Tonge.

A word of caution. In writing these notes I have had in mind my students more than my colleagues. Thus, the result has been more an introduction than a survey. The experts will find some of their theorems missing, their proofs changed, their points of view distorted. The choice of material has been personal, and probably guided by aesthetic sense more than anything else. Also, in many places, the absence of a reference is not due to my originality, but to my ignorance.

Before going on to the first section of this paper, we succintly present the basic definitions and notation of the theory of polynomials on Banach spaces. For a truly comprehensive text, see [D4].

A k-homogeneous polynomial $P : E \longrightarrow F$ is a function which can be written as $P(x) = A(x, \dots, x)$ where A is a k-linear function from $E \times \cdots \times E$ to F. When A is symmetric, it can be recovered from the corresponding polynomial P by any of several polarization formulas. Thus, there is a one-to-one correspondence between k-homogeneous polynomials and symmetric k-linear functions.

The 'uniform' norm of P can be defined as

$$\|P\| = \sup\{\|P(x)\| : x \in B_E\}$$

where B_E is the closed unit ball of E. Continuity of P (and of the corresponding symmetric k-linear function A) is equivalent to finiteness of $\|P\|$. The space of F-valued continuous k-homogeneous polynomials, with the uniform norm, is a Banach space which we will denote by $P(^kE, F)$, or simply by $P(^kE)$ when F is the scalar field.

We will need to consider several more restricted classes of scalar-valued polynomials, each of which gives rise to a vector subspace of $P(^kE)$.

Surely the simplest kind of polynomials over a Banach space are the polynomials of *finite type*, those which may be written as $P(x) = \sum_{i=1}^{n} \gamma_i(x)^k$, where each γ_i is a continuous linear form over E. Slightly larger is the class of *nuclear* polynomials, made up of those which may be represented (non-uniquely) as $P(x) = \sum_{i=1}^{\infty} \gamma_i(x)^k$, with $\sum_{i=1}^{\infty} \|\gamma_i\|^k < \infty$. These form a non-closed vector subspace of $P(^kE)$ (although they do form a Banach space in the 'nuclear' norm, i.e. the infimum of $\sum_{i=1}^{\infty} \|\gamma_i\|^k$ over all possible representations of P). The closure of the space of nuclear (or also of the finite-type polynomials) in the uniform norm is the space of *approximable* polynomials. Larger still are the spaces of *weakly continuous* polynomials (those which are weakly continuous over all bounded subsets of E) and *weakly sequentially continuous* polynomials. We thus have the chain of vector subspaces of $P(^kE)$

$$P_f(^kE) \subset P_N(^kE) \subset P_A(^kE) \subset P_w(^kE) \subset P_{wsc}(^kE) \subset P(^kE).$$

Reverse inclusions sometimes hold, highlighting the interplay between the study of polynomials and Banach space theory: if E' has the approximation property $P_A(^kE) = P_w(^kE)$ [AP], if E has the Dunford-Pettis property $P_{wsc}(^kE) = P(^kE)$ [Ry1], while $P_w(^kE) = P_{wsc}(^kE)$ holds if and only if E contains no copy of ℓ^1 [FGLl]. Another important class of polynomials is that of *integral* polynomials: those which may be represented by a regular Borel measure μ over the unit ball of E' (considered with the weak* topology) in the following way

$$P(x) = \int_{B_{E'}} \gamma(x)^k d\mu(\gamma).$$

Like the nuclear polynomials, these form a Banach space with their own particular 'integral' norm (the infimum of the total variations of the representing measures), but are a non-closed subspace of $P(^kE)$ in the uniform norm. We have the inclusions

$$P_N(^kE) \subset P_I(^kE) \subset P_{wsc}(^kE),$$

with $P_N(^kE) = P_I(^kE)$ when E' has the Radon-Nikodym property [A] (for example, if E is reflexive).

The only trivially extendible scalar-valued polynomials are the finite-type and the nuclear polynomials. These can be extended to any larger space by simply using the Hahn-Banach theorem to extend their component linear forms γ_i. We shall see that the integral polynomials are also extendible but that there are, in general, approximable polynomials which cannot be extended.

§1. The Aron-Berner extension

In 1951 R. Arens ([A1], [A2]) found a way of extending the product of a Banach algebra E to its bidual E'' in such a way that this bidual became itself a Banach algebra. We restrict our presentation here to commutative Banach algebras E, for in the sequel we will be interested in polynomials, and thus in symmetric multilinear forms.

Suppose then that $a, b \in E$, $S, T \in E''$, and $\gamma \in E'$. The elements $(a\gamma)$ and $T_\gamma \in E'$ are defined by setting

$$(a\gamma)(b) = \gamma(ab) \qquad \text{and} \qquad T_\gamma(a) = T(a\gamma).$$

Now, given S and T in the bidual of E, define the product of S and T by

$$(ST)(\gamma) = S(T_\gamma)$$

for each $\gamma \in E'$. It is easily seen that this product extends the product in E. Also, we observe the following two facts which will have analogues in the more general Aron-Berner extension.

i) The map $(S, T) \mapsto (ST)(\gamma)$ is weak*-continuous of its first variable: Indeed, if $(S_i) \subset E''$ is weak*-converging to S, then $(S_iT)(\gamma) = S_i(T_\gamma) \longrightarrow S(T_\gamma) = (ST)(\gamma)$.

ii) All elements of E commute with all elements of E'': Take $a \in E$ and $T \in E''$. Then for any γ

$$(aT)(\gamma) = a(T_\gamma) = T_\gamma(a) = T(a\gamma)$$
$$(Ta)(\gamma) = T(a_\gamma)$$

but one checks that a_γ is just $(a\gamma)$.

In general, the map in i) is not continuous of the second variable, and the Arens product is not commutative. In fact we will see in the next section that these two properties are equivalent. To see that the product is not commutative, we present the following example [Z2] (see also [Re]).

Example 1.1.

Let E be the commutative Banach algebra $\ell^1(Z)$ with the convolution product

$$(ab)_j = \sum_{i \in Z} a_i b_{j-i},$$

and define the subspace V of $E' = \ell^\infty(Z)$

$$V = \{\gamma \in E' : L^-(\gamma) = \lim_{n \to -\infty} \gamma_n \text{ and } L^+(\gamma) = \lim_{n \to +\infty} \gamma_n \text{ exist}\}.$$

Now use Hahn-Banach to extend L^- and L^+ to elements S^- and S^+ of E''. These do not commute. To see this, take $\gamma = (\ldots, -1, -1, 0, 1, 1, \ldots) \in E'$ and observe that

$$(a\gamma)_n = \sum_k a_k \gamma_{n+k} = \sum_{k > -n} a_k - \sum_{k < -n} a_k.$$

Thus

$$S^-_\gamma(a) = \lim_{n \to -\infty} \Big(\sum_{k > -n} a_k - \sum_{k < -n} a_k \Big) = -\sum_k a_k,$$

and $S_\gamma^- = (\ldots, -1, -1, -1, \ldots)$. Analogously $S_\gamma^+ = (\ldots, 1, 1, 1, \ldots)$. Now

$$(S^+S^-)(\gamma) = S^+(S_\gamma^-) = -1 \neq 1 = S^-(S_\gamma^+) = (S^-S^+)(\gamma).$$

For more on the Arens product, see [IL], [DH], and [GI].

The process applied to the map $(a, b) \mapsto \gamma(ab)$ can be carried out on an arbitrary continuous bilinear function

$$X \times Y \longrightarrow Z$$

as follows: fix $z' \in Z'$ and $x \in X$; then we obtain an element of Y'. This mapping is bilinear and continuous, producing $Z' \times X \longrightarrow Y'$. If we repeat the process again we obtain $Y'' \times Z' \longrightarrow X'$, and when repeated once more,

$$X'' \times Y'' \longrightarrow Z''.$$

Now if instead of considering Banach algebras one takes $X = Y = E$, a Banach space, $F = C$, and $A : E \times E \longrightarrow C$ a symmetric bilinear form on E, the construction produces an extension of A to E''.

The same extension can be defined in other ways. Here are two:
a) Say z_1 and z_2 are elements of the bidual, and choose nets (x_i) and (y_i) converging to z_1 and z_2 in the weak* topology of E''. Now put

$$\overline{A}(z_1, z_2) = \lim_i \lim_j A(x_i, y_j).$$

This definition does not depend on the particular choice of nets, and produces the same extension as above. Note that $\overline{A}$ is not, in general, weak*-continuous in the second variable, but only in the first.
b) Define the associated symmetric linear map $T : E \longrightarrow E'$ by $T(x)(y) = A(x, y)$. Its bitranspose $T'' : E'' \longrightarrow E'''$ corresponds to a continuous bilinear extension of A

$$\overline{A}(z_1, z_2) = T''(z_1)(z_2).$$

Similar constructions can be carried out for k-linear forms. Note that in all cases there is a choice involved in the ordering (of the limits, of z_1 and z_2). It turns out that even though A is symmetric, the extended form $\overline{A}$ need not be symmetric. In fact one has the following equivalences, with A and T as above.

Proposition 1.2. *([DH], [Rp]) The following are equivalent:*
i) $\overline{A}$ *is symmetric.*
ii) $\overline{A}$ *is separately weak*-continuous in both variables.*
iii) $T : E \longrightarrow E'$ *is weakly compact.*

Proof. i) implies ii): $\overline{A}(z_1, z_2) = T''(z_1)(z_2)$ is always weak*-continuous in the first variable, for transpose maps are weak* to weak*-continuous. Thus symmetry implies weak*-continuity in the other variable as well.

ii) implies iii): Fix any $z_1 \in E''$. Weak*-continuity of $T''(z_1)(z_2)$ in the variable z_2 says that $T''(z_1)$ is actually in the canonical inclusion of E' in E'''. Thus $T''(E'') \subset E'$ and T is weakly compact by Gantmacher's theorem [HP].

iii) implies i): We denote the canonical inclusion of E in E'' by J_E and that of E' in E''' by $J_{E'}$. First, note that since T is symmetric, $T' \circ J_E = T$. Indeed, for

any x and y in E,

$$\begin{aligned}(T' \circ J_E)(x)(y) =& T'(J_E(x))(y) = (J_E(x) \circ T)(y)\\ =& J_E(x)(T(y)) = T(y)(x)\\ =& T(x)(y)\end{aligned}$$

This in turn implies that for any $z \in E''$, $T'(z) = T''(z) \circ J_E$: for all $x \in E$, we have

$$\begin{aligned}T'(z)(x) =& (z \circ T)(x) = (z \circ T' \circ J_E)(x)\\ =& (z \circ T')(J_E(x)) = (T''(z))(J_E(x))\\ =& (T''(z) \circ J_E)(x)\end{aligned}$$

Thus, if T is weakly compact we obtain $T'' = J_{E'} \circ T'$: Since $T''(E'') \subset J_{E'}(E')$, and E separates points of E', we need only see that for $z \in E''$ and $x \in E$, $T''(z)(J_E(x)) = T'(z)(x)$. But this is $T'(z) = T''(z) \circ J_E$. Thus we have,

$$\begin{aligned}T''(z_1)(z_2) =& (J_{E'} \circ T')(z_1)(z_2) = J_{E'}(T'(z_1))(z_2)\\ =& z_2(T'(z_1)) = (z_2 \circ T')(z_1)\\ =& T''(z_2)(z_1)\end{aligned}$$

and T'' is symmetric. □

It is worth noting that elements of E always commute with those of E'', even when $\overline{A}$ is not symmetric. Thus $\overline{A}(x, z) = \overline{A}(z, x)$ for any choice of $x \in E$ and $z \in E''$.

If the equivalent conditions in the proposition hold for all symmetric A, E is said to be *symmetrically regular*, and if all (symmetric or otherwise) operators $T : E \longrightarrow E'$ are weakly compact, E is *regular*. As Example 1.1 shows, $\ell^1(Z)$ is not symmetrically regular (and hence not regular). The operator $T : \ell^1(Z) \longrightarrow \ell^\infty(Z)$ corresponding to Example 1.1 is given by

$$T(x)_i = \sum_j \operatorname{sgn}(j) x_{j-i}$$

where the sign of 0 is taken as 0. It is easily seen that T is not weakly compact: any subsequence of $T(e_n) = (\ldots, -1, -1, 0, 1, 1, \ldots)$ (with 0 in $-n$th place) which converges weakly does so to $(\ldots, 1, 1, 1, \ldots)$, but $S^-(T(e_n)) = -1$ for all n. So the image of the unit ball is not weakly sequentially compact.

Another example of such an operator $T : \ell^1 \longrightarrow \ell^\infty$ [ACG1] is given by

$$T(x)_i = \sum_{j \text{even}; 1 \le k < j} x_j \delta_{ik} + \sum_{k \text{even}; 1 \le j < k} x_j \delta_{ik}$$

and yet another [Re] by

$$T(x)_i = (-1)^{i+1} \sum_{j \le i} x_j + \sum_{j > i} (-1)^{j+1} x_j.$$

On the other hand, $C(K)$ spaces are regular [GI], as are all C^*-algebras.

We continue with some results on symmetric regularity, as found in [AGGM].

Proposition 1.3. *$E\times F$ is regular if and only if all operators in $L(E,E')$, $L(E,F')$, $L(F,E')$, $L(F,F')$ are weakly compact.*

Proof. Just note that one may write $T : E \times F \longrightarrow E' \times F'$ as

$$T = \begin{pmatrix} R_1 & R_2 \\ R_3 & R_4 \end{pmatrix},$$

thus $T(x,y) = (R_1(x) + R_2(y), R_3(x) + R_4(y))$. □

An immediate consequence is that E is regular if and only if $E \times E$ is regular. Also, if E is non-reflexive, $E \times E'$ is non-regular.

Proposition 1.4.
i) If $E \times E$ is symmetrically regular, then E is regular.
ii) $E \times E$ is regular if and only if it is symmetrically regular.

Proof. i) Take $T : E \longrightarrow E'$, any continuous operator, and define $S : E \times E \longrightarrow E' \times E'$ by

$$S = \begin{pmatrix} 0 & T_0 \\ T & 0 \end{pmatrix},$$

where $T_0(y)(u) = T(u)(y)$. Then

$$\begin{aligned} S(x,y)(u,v) &= (T_0(y), T(x))(u,v) = T_0(y)(u) + T(x)(v) \\ &= T(u)(y) + T(x)(v) = T(x)(v) + T(u)(y) \\ &= T_0(v)(x) + T(u)(y) = (T_0(v), T(u))(x,y) \\ &= S(u,v)(x,y) \end{aligned}$$

Thus S is symmetric, and therefore it, and T, are compact.

ii) This is now easy from part i) and the fact that E is regular if and only if $E \times E$ is. □

We now define the Aron-Berner extension of a continuous k-homogeneous scalar-valued polynomial P defined over the Banach space E. We choose to give a degree-k version of the first definition mentioned above, but note that the second is also easily adapted to this case. We first extend k-linear forms:

Given $z \in E''$ and $1 \leq m \leq k$, define

$$\overline{z} : L_s(^m E) \longrightarrow L_s(^{m-1} E)$$

by $\overline{z}(A)(x_1, \ldots, x_{m-1}) = z(A_{x_1 \ldots x_{m-1}})$. Here $A_{x_1 \ldots x_{m-1}}$ denotes the linear form obtained from A by fixing $m-1$ variables (note that by symmetry it does not matter where these variables are fixed). The map $\overline{z}$ is linear, continuous and of norm $\|z\|$. Now, given $A \in L_s(^k E)$ we may define the extended k-linear form $\overline{A} \in L(^k E'')$ by

$$\overline{A}(z_1, \ldots, z_k) = \overline{z}_1 \circ \cdots \circ \overline{z}_k(A).$$

Here we have identified $L_s(^0 E)$ with the scalar field. The extended mapping $\overline{A}$ need not be symmetric. We have in fact $k!$ extensions, one for each choice of ordering in $\overline{z}_1 \circ \cdots \circ \overline{z}_k$. Note however that for $x \in E$ we do have $\overline{z} \circ \overline{x} = \overline{x} \circ \overline{z}$,

and that $\overline{A}$ is weak*-continuous in the first variable [Z1]. These properties are analogous to those of the Arens product.

When E is symmetrically regular, variables in A can be permuted in pairs even in the extensions of k-linear forms, so that if E is symmetrically regular the extension of any symmetric k-linear form is itself symmetric.

We define the *Aron-Berner extension* of P to the bidual E'' of E by setting

$$\overline{P}(z) = \overline{A}(z, \dots, z),$$

where A is the unique symmetric continuous k-linear form $A : E \times \cdots \times E \longrightarrow C$ such that $P(x) = A(x, \dots, x)$ for every $x \in E$.

It must be stressed that this extension is quite algebraic in nature. Indeed, the formal manipulations involved can be carried out with no regard to topology whatsoever. It is certainly not an extension by continuity, and $\overline{P}$ need not be weak*-continuous (consider the polynomial $P(x) = \sum_n x_n^2$ over ℓ^2).

However the Aron-Berner extension respects most of the usual classes of continuous polynomials, as we shall see below. Also, a relation with the weak*-topology in E'' is present, though perhaps more subtly than one would expect, as the next two theorems show.

First, there is the following characterization of the Aron-Berner extension [Z1] in terms of first-order differentials.

Theorem 1.5. *[Z1] If $Q \in P(^kE'')$ and $P = Q|_E$, then $Q = \overline{P}$ if and only if*
a) For each $x \in E$, $DQ(x)$ is weak*-continuous, and
b) For each $z \in E''$ and $(x_\alpha) \subset E$ converging weak* to z, $DQ(z)(x_\alpha) \to DQ(z)(z)$. □

Condition b) is somewhat odd, but examples show that $D\overline{P}(z)$ need not be weak*-continuous for $z \in E''$ and that condition a) alone does not guarantee that $Q = \overline{P}$. Indeed, let $E = \ell^1(Z) \times \ell^1(Z)$ and $P : E \longrightarrow C$ such that $P(a, b) = \gamma(ab)$ (with notation as in Example 1.1). Then the Aron-Berner extension of P is $\overline{P}(S, T) = \frac{1}{2}[ST + TS](\gamma)$. However, since $(T, S) \mapsto (TS)(\gamma)$ is not weak*-continuous in the second variable, choose (S_α) converging weak* to S with $(TS_\alpha)(\gamma) \not\to (TS)(\gamma)$; let $z = (S, T)$ and $z_\alpha = (S_\alpha, T)$. Then $z_\alpha \to z$ weak*, but $D\overline{P}(z)(z_\alpha) \not\to D\overline{P}(z)(z)$. Also, if $Q(S, T) = (ST)(\gamma)$, then $DQ(x)$ is weak*-continuous for all $x \in E$, but $Q \neq \overline{P}$.

One consequence of the above characterization is the linearity of the Aron-Berner extension mapping $P \longrightarrow \overline{P}$. Indeed, $\overline{P} + \overline{Q}|_E = P + Q$, and for all $z \in E''$ the differential $D(\overline{P} + \overline{Q})(z) = D\overline{P}(z) + D\overline{Q}(z)$ has the same continuity properties as each summand. The same holds for scalar multiples of $\overline{P}$.

Thus, the Aron-Berner extension is a one-to-one linear map

$$P(^kE) \longrightarrow P(^kE'')$$

The continuity of this map is a simple matter, for the norm of the extended polynomial verifies: $\|\overline{P}\| \leq \|\overline{A}\| = \|A\| \leq \frac{k^k}{k!}\|P\|$. However, more can be said. Davie and Gamelin have proved [DG] the following theorem which shows that in fact the Aron-Berner map is an isometry onto its image.

Theorem 1.6. $\|\overline{P}\| = \|P\|$. *In fact, given $z \in E''$ and a polynomial P, there is a bounded net $(x_\alpha) \subset E$ weak*-converging to z and for which $P(x_\alpha) \to \overline{P}(z)$.*

Proof. Let A be the symmetric k-linear form associated with P. Fix z in the unit ball of E'', and $\varepsilon > 0$. For any natural number N (to be fixed later) choose, using Goldstine's theorem, an element x_1 in the unit ball of E such that

$$|\overline{A}(z, z, \ldots, z) - \overline{A}(x_1, z, \ldots, z)| < \frac{\varepsilon}{k}.$$

Then choose x_2 in the unit ball of E such that

$$\begin{aligned} |\overline{A}(z, z, \ldots, z) - \overline{A}(x_2, z, \ldots, z)| <& \frac{\varepsilon}{k} \\ \text{and } |\overline{A}(x_1, x_2, z, \ldots, z) - \overline{A}(x_1, z, z, \ldots, z)| <& \frac{\varepsilon}{k} \end{aligned}$$

and go on until one has $x_1, \ldots, x_N$ in the unit ball of E such that whenever $i_1 < \cdots < i_r$, then

$$|\overline{A}(x_{i_1}, \ldots, x_{i_{r-1}}, x_{i_r}, z, \ldots, z) - \overline{A}(x_{i_1}, \ldots, x_{i_{r-1}}, z, z, \ldots, z)| < \frac{\varepsilon}{k}.$$

We then have, for $i_1 < \cdots < i_k$, or (considering that A is symmetric) for any k different indexes $i_1, \ldots, i_k$

$$\begin{aligned} |\overline{A}(z, \ldots, z) - A(x_{i_1}, \ldots, x_{i_k})| \leq& |\overline{A}(z, z, \ldots, z) - \overline{A}(x_{i_1}, z, \ldots, z)| \\ &+ |\overline{A}(x_{i_1}, z, \ldots, z) - \overline{A}(x_{i_1}, x_{i_2}, z, \ldots, z)| \\ &+ \cdots \\ &+ |\overline{A}(x_{i_1}, \ldots, x_{i_{k-1}}, z) - A(x_{i_1}, \ldots, x_{i_k})| < \varepsilon. \end{aligned}$$

Now let $x = \frac{1}{N} \sum_{i=1}^{N} x_i$.

$$\begin{aligned} |\overline{P}(z) - P(x)| =& \left| \overline{A}(z, \ldots, z) - A\left(\frac{1}{N} \sum_{i_1=1}^{N} x_{i_1}, \ldots, \frac{1}{N} \sum_{i_k=1}^{N} x_{i_k} \right) \right| \\ =& \left| \overline{A}(z, \ldots, z) - \frac{1}{N^k} \sum_{i_1=1}^{N} \cdots \sum_{i_k=1}^{N} A(x_{i_1}, \ldots, x_{i_k}) \right| \\ \leq& \frac{1}{N^k} \sum_{i_1=1}^{N} \cdots \sum_{i_k=1}^{N} |\overline{A}(z, \ldots, z) - A(x_{i_1}, \ldots, x_{i_k})| \\ =& \sum_1 + \sum_2, \end{aligned}$$

where in the first sum we group all terms with no repeated indexes, and in the second all remaining terms. Since each term with non-repeated indexes is smaller

than ε, we have

$$\begin{aligned}\sum_1 + \sum_2 \leq & \varepsilon + \sum_2 \\ \leq & \varepsilon + 2\|A\| \frac{1}{N^k} \left[\text{number of terms in } \sum_2 \right] \\ = & \varepsilon + 2\|A\| \frac{1}{N^k} [N^k - N(N-1)\cdots(N-k+1)] \\ = & \varepsilon + 2\|A\| \left(1 - \frac{N}{N}\frac{N-1}{N}\cdots\frac{N-k+1}{N}\right)\end{aligned}$$

but the term on the right can be made small by taking N large; i.e. the larger the number of indexes $i_1, \ldots, i_N$ that one uses, the smaller the chance of having repeated indexes in a group of k of them. Thus $P(x)$ can be as close as $\overline{P}(z)$ as required. □

A polynomial P may in general admit many other norm-preserving extensions to E'', but there are a few uniqueness results. For example, 2-homogeneous norm-attaining polynomials on c_0 have unique norm-preserving extension to ℓ^∞ [ABC].

In view of the Davie-Gamelin theorem, one can consider the Aron-Berner extension as an inclusion map

$$P(^kE) \longrightarrow P(^kE'').$$

A few comments on the preservation of the usual classes of polynomials through the Aron-Berner extension: Clearly finite type, nuclear, and approximable polynomials are preserved. Less clear but nevertheless true is the fact that integral polynomials over E are mapped into integral polynomials over E''. We have:

Theorem 1.7. *[CZ1] The Aron-Berner extension of an integral polynomial over E is an integral polynomial over E''. Furthermore, if E contains no copy of ℓ^1, and μ is a measure representing P, then*

$$\overline{P}(z) = \int_{B_{E'}} z(\gamma)^k d\mu(\gamma). \quad \square$$

Note that this integral makes no sense if E contains a copy of ℓ^1, for z need not be a μ-measurable function over $B_{E'}$ [H]. Also, an analogue of the Davie-Gamelin theorem holds: the integral norm of P is the same as that of $\overline{P}$. The same holds for the class of extendible polynomials (which we will present in §3). De Moraes has proved:

Theorem 1.8. *[Mo1] The Aron-Berner extension of a weakly continuous polynomial over E is a weakly continuous polynomial over E''.* □

Another proof of this fact is possible through the characterization of weakly continuous polynomials as those which are continuous with respect to the seminorm

$$\|x\|_K = \sup\{|\gamma(x)| : \gamma \in K\}$$

for some compact subset K of E' [ALRR], [T]. Indeed, it has been shown [AG], [CDDL] that for any particular K, such polynomials are preserved by the Aron-Berner extension.

The Aron-Berner extension does not, in general, preserve the class P^k_{wsc} of weakly sequentially continuous polynomials, as the following example shows.

Example 1.9. Given any Banach space X, consider the 2-homogeneous polynomial $X \times X' \longrightarrow C$ defined by $(x, \gamma) \mapsto \gamma(x)$. It is easily seen that weak sequential continuity of this polynomial is equivalent to X having the Dunford-Pettis property.

Now consider a Banach space E having the Dunford-Pettis property whose dual E' does not have it (say $(\sum_n \ell^{2^n})_1$, [S]), and consider the weakly sequentially continuous polynomial

$$P : E \times E' \longrightarrow C, \text{ defined by } P(x, \gamma) = \gamma(x).$$

The Aron-Berner extension of this polynomial is

$$\overline{P} : E'' \times E''' \longrightarrow C, \text{ given by } \overline{P}(z, \alpha) = \frac{\alpha(z) + z(r(\alpha))}{2},$$

where $r : E''' \longrightarrow E'$ is the restriction mapping (i. e., $r = J'_E$). Thus if we restrict $\overline{P}$ to the subspace $E'' \times E'$, we obtain the polynomial

$$Q : E'' \times E' \longrightarrow C \text{ given by } Q(z, \gamma) = z(\gamma).$$

If $\overline{P}$ were weakly sequentially continuous, Q would be also, and hence E' would have the Dunford-Pettis property, contradicting our assumption.

A positive result in this matter is the following theorem of González and Gutiérrez (who actually prove a more general result).

Theorem 1.10. *[GG] If E' has the Dunford-Pettis property, then the Aron-Berner extension preserves weakly sequentially continuous polynomials.* □

Through the Aron-Berner extension, polynomials defined on E can be evaluated at points of E'', and indeed at points of any even dual (E'', E^{iv},...) of E. It is natural to ask then if evaluations at such points constitute new evaluations or if, for example, given $w \in E^{iv}$ there is an element $z \in E''$ which polynomials $P \in P(^kE)$ cannot distinguish from w, that is $\overline{P}(z) = \overline{\overline{P}}(w)$ for all such P. Clearly, for $k = 1$ there is such an element of E''; it is $J'_{E'}(w) = w \circ J_{E'}$, where $J_{E'}$ is the canonical inclusion of E' in its bidual E'''. The matter for $k > 1$ is far more interesting, and has been studied by Aron, Galindo, García and Maestre, who found the following.

Theorem 1.11. *[AGGM] Each evaluation at a point of E^{iv} is in fact an evaluation at a point of E'' if and only if E is symmetrically regular.* □

For F-valued polynomials $P : E \longrightarrow F$, the extension process described above produces an F''-valued polynomial $\overline{P} : E'' \longrightarrow F''$. It can be seen that if P is weakly compact, then the image of $\overline{P}$ is in F, although (unlike the linear case) the converse is not true. However, even in situations in which $\overline{P}$ is F''-valued, analogues of Theorem 1.5 and Theorem 1.6 hold [C1].

§2. Extension Morphisms from E to G

Several particular cases have been studied ([AB], [DT], [GGMM], [LR], [Z1]) of situations in which all polynomials defined over a Banach space E are extended to a certain larger Banach space G. In all these instances there turns out to be an extension morphism. We present in this section a unified version of the above mentioned results, and several formulations equivalent to the existence of an extension morphism for polynomials.

We shall need the following result of Aron and Schottenloher [AS].

Theorem 2.1. *Let E and F be arbitrary Banach spaces. Then $L(E,F)$ is isomorphic to a complemented subspace of $P(^kE,F)$.*

Proof. Fix $e \in E$ and $\alpha \in E'$ a norm-one linear functional such that $\alpha(e) = 1$. Define the 'inclusion' $\rho : L(E,F) \longrightarrow P(^kE,F)$ by $\rho(T) = T \cdot \alpha^{k-1}$ (the product is point-wise). Also, define the 'projection' $\pi : P(^kE,F) \longrightarrow L(E,F)$ by $\pi(Q) = DQ(e) - (k-1)Q(e) \cdot \alpha$. Note that $DQ(e)(x) = kA(e,\dots,e,x)$, where A is the symmetric k-linear operator associated to Q, and that if $Q = T \cdot \alpha^{k-1}$,

$$A(e,\dots,e,x) = \frac{1}{k}\left[T(e)\alpha(e)\cdots\alpha(x) + \cdots + \alpha(e)\cdots T(e)\alpha(x) + \alpha(e)\cdots\alpha(e)T(x)\right].$$

Thus $(\pi \circ \rho)(T) = T$. Indeed, for any $x \in E$, we have

$$\begin{aligned}\pi(\rho(T))(x) &= kA(e,\dots,e,x) - (k-1)T(e)\alpha(x)\\ &= (k-1)T(e)\alpha(x) + T(x) - (k-1)T(e)\alpha(x)\\ &= T(x).\end{aligned}$$

□

The existence of a linear extension mapping $P(^kE,F) \longrightarrow P(^kG,F)$ is equivalent to the existence of a linear extension mapping $L(E,F) \longrightarrow L(G,F)$. We have the following theorem. The 'if' part is a reformulation of results in [GGMM] and [Ni], the 'only if' part is due to A. Peris.

Theorem 2.2. *Let E be a subspace of G, and F an arbitrary Banach space. Then there is an extension morphism $\overline{s} : P(^kE,F) \longrightarrow P(^kG,F)$ if and only if there is an extension morphism $s : L(E,F) \longrightarrow L(G,F)$*

Proof. Suppose $\overline{s}$ exists, and define ρ and π as in Theorem 2.1. Now let $s = \pi \circ \overline{s} \circ \rho$. s is continuous, linear, and for $T \in L(E, F)$ and $x \in E$, $s(T)(x) = T(x)$:

$$\begin{aligned} s(T)(x) &= \pi(\overline{s}(\rho(T)))(x) \\ &= D[\overline{s}(\rho(T))](e)(x) - (k-1)\overline{s}(\rho(T))(e)\alpha(x) \\ &= D[\rho(T)](e)(x) - (k-1)\rho(T)(e)\alpha(x) \\ &= T(x), \end{aligned}$$

(the last equality as in Theorem 2.1). Note also that in differentiating, we have used the fact that $e + \lambda x$ is in E for any λ.

Now suppose that s is given. We want to define $\overline{s}$, and proceed as in the definition of the Aron-Berner extension. Given $g \in G$ and $1 \leq m \leq k$, define

$$\overline{g} : L_s(^m E, F) \longrightarrow L_s(^{m-1} E, F)$$

by $\overline{g}(A)(x_1, \ldots, x_{m-1}) = s(A_{x_1 \ldots x_{m-1}})(g)$. Here $A_{x_1 \ldots x_{m-1}}$ denotes the linear form obtained from A by fixing $m-1$ variables. Now, given $A \in L_s(^k E, F)$ we may define the extended k-linear function $\overline{A} \in L(^k G, F)$ by

$$\overline{A}(g_1, \ldots, g_k) = \overline{g}_1 \circ \cdots \circ \overline{g}_k(A).$$

Thus if P is a polynomial in $P(^k E, F)$, and A its associated symmetric k-linear function, set

$$\overline{s}(P)(g) = \overline{A}(g, \ldots, g).$$

It is now easily checked that $\overline{s}$ is a linear extension mapping from $P(^k E, F)$ to $P(^k G, F)$. □

Note that the function $\overline{s}$ cannot be defined (even non-linearly) if s is not linear. Indeed, a non-linear s would induce a $\overline{g}(A)$ which would fail to be $(m-1)$-linear in the above definition. Also, note that the constructions $s \mapsto \overline{s}$ and $\overline{s} \mapsto s$ are not in general inverse of each other.

The fact that E is a subspace of G and s an extension map makes $\overline{s}$ an extension map; however, the construction can be carried through for any continuous linear map $t : L(E, F) \longrightarrow L(G, F)$, giving rise to a continuous linear map

$$\overline{t} : P(^k E, F) \longrightarrow P(^k G, F).$$

Note however, that $\overline{t \circ s}$ need not be $\overline{t} \circ \overline{s}$. The following example is taken from [LZ].

Example 2.3. $\overline{t \circ s} \neq \overline{t} \circ \overline{s}$:

If X is a Banach space, consider (as in Example 1.9) the 2-homogeneous polynomial P defined over $X \times X'$ by $P(x, x') = x'(x)$. Recall that the Aron-Berner extension of P to $X'' \times X'''$ is

$$\overline{P}(x'', x''') = \frac{1}{2}[x'''(x'') + x''(\rho(x'''))],$$

where $\rho : X''' \to X'$ is the restriction (i.e., the transpose of the natural inclusion $J_X : X \to X''$). We will use the notation

$$E = X \times X' \qquad F = X'' \times X' \qquad G = X'' \times X'''$$

and over these spaces consider the polynomials

$$P(x,x') = x'(x) \qquad Q(x'',x') = x''(x')$$

and morphisms $s : E' \to F'$ and $t : F' \to G'$ given by $s = J_{X'} \oplus id_{X''}$,$t = id_{X'''} \oplus J_{X''}$. Also, we denote $r = J'_{X'}$. We now check that $\overline{t \circ s} \neq \overline{t} \circ \overline{s}$.

First, calculate $\overline{\overline{s}(P)}$:

$$\overline{\overline{s}(P)}(x^{iv}, x''') = \overline{Q}(x^{iv}, x''') = \frac{1}{2}[x^{iv}(x''') + x'''(r(x^{iv}))].$$

Now calculate $t' \circ J_G$:

$$\begin{aligned}(t' \circ J_G)(x'', x''') &= t'(J_{X''}(x''), J_{X'''}(x''')) \\ &= (J_{X''}(x''), J'_{X''}(J_{X'''}(x'''))) \\ &= (J_{X''}(x''), x'''),\end{aligned}$$

so using this we have

$$\begin{aligned}(\overline{t} \circ \overline{s})(P)(x'', x''') &= (\overline{\overline{s}(P)} \circ t' \circ J_G)(x'', x''') \\ &= \overline{\overline{s}(P)}(J_{X''}(x''), x''') \\ &= \frac{1}{2}[J_{X''}(x'')(x''') + x'''(r(J_{X''}(x'')))] \\ &= x'''(x'').\end{aligned}$$

Then

$$\begin{aligned}(\overline{t \circ s})(P)(x'', x''') &= (\overline{P} \circ (t \circ s)' \circ J_G)(x'', x''') \\ &= (\overline{P} \circ s')((t' \circ J_G)(x'', x''')) \\ &= (\overline{P} \circ s')(J_{X''}(x''), x''') \\ &= \overline{P}(r(J_{X''}(x'')), x''') \\ &= \overline{P}(x'', x''') \\ &= \frac{1}{2}[x'''(x'') + x''(\rho(x'''))],\end{aligned}$$

thus $\overline{t \circ s} = \overline{t} \circ \overline{s}$ if and only if $x''(\rho(x''')) = x'''(x'')$, but this only happens for reflexive X.

We now provide some examples of situations in which a linear extension map exists for linear operators, and thus also for polynomials. We begin with the case of F-valued polynomials in the first three examples, and continue with scalar-valued polynomials in the rest.

Example 2.4. *(E complemented in G):*

This is clearly the simplest case for extension. If $p : G \longrightarrow E$ is a projector, any polynomial Q may be extended by setting $\overline{s}(Q) = Q \circ p$. This extension is a morphism and comes from $s : L(E,F) \longrightarrow L(G,F)$ $(s(T) = T \circ p)$ by the construction above.

Example 2.5. *([Z1]):*

If $G = L(L(E,F),F)$, E may be considered contained in G through the inclusion mapping $J : E \longrightarrow G$ defined by $J(x)(T) = T(x)$. In this case one may define $s : L(E,F) \longrightarrow L(G,F)$ by $s(T)(g) = g(T)$.

Example 2.6.

If there is a linear extension map for linear forms $\sigma : E' \longrightarrow G'$, and F is complemented in its bidual (this happens, for example if F is a dual space), one may define $s : L(E,F) \longrightarrow L(G,F)$ as follows. First define $s_0 : L(E,F) \longrightarrow L(G,F'')$ by $s_0(T)(g)(f') = \sigma(f' \circ T)(g)$, and then $s(T) = p \circ s_0(T)$, where p is the projection $F'' \longrightarrow F$. If σ exists but F is not complemented in F'', one may define an extension morphism $P(^kE,F) \longrightarrow P(^kG,F'')$.

Example 2.7. *(the Aron-Berner extension [AB]):*

For $G = E''$, the Aron-Berner extension comes from the extension morphism $s : E' \longrightarrow E'''$, the canonical inclusion of E' in its bidual.

Example 2.8. *(G in an 'local' ultrapower of E [DT], [LR]):*

Suppose $E \subset G$ and G is finitely represented in E in the following sense: for each finite-dimensional subspace of G and each $\varepsilon > 0$ there is a isomorphism $T : S \longrightarrow T(S) \subset E$ such that T is the identity when restricted to $S \cap E$ and $\|T\|\|T^{-1}\| < 1+\varepsilon$. In such a situation one can define a linear extension morphism $s : E' \longrightarrow G'$ in the following way: Consider the indexing set $I = \{(S,\varepsilon) : \dim S < \infty, \varepsilon > 0\}$ ordered by $(S_1,\varepsilon_1) \leq (S_2,\varepsilon_2) \Leftrightarrow S_1 \subset S_2$ and $\varepsilon_2 \leq \varepsilon_1$. The sets $A_i = \{j \in I : j \geq i\}$ form a filter basis over I. Let $\mathcal{U}$ be any ultrafilter containing this basis. We have, for any $i = (S,\varepsilon) \in I$,

$$T_i : S \longrightarrow T_i(S) \subset E$$

such that $\|T_i\|\|T_i^{-1}\| < 1+\varepsilon$. For each $\gamma \in E'$, define

$$\gamma_i(g) = \begin{cases} \gamma(T_i(g)), & \text{if } g \in S \\ 0, & \text{if } g \notin S. \end{cases}$$

Although the γ_i's are not linear, $\lim_{\mathcal{U}} \gamma_i(g)$ is, and we may define

$$s : E' \longrightarrow G' \qquad \text{by } s(\gamma)(g) = \lim_{\mathcal{U}} \gamma_i(g).$$

s is then a linear extension map, for if $x \in E$, $s(\gamma)(x) = \lim_{\mathcal{U}} \gamma(T_i(x)) = \gamma(x)$. This induces an extension morphism $\overline{s} : P(^kE) \longrightarrow P(^kG)$, which is the extension $P \mapsto \widehat{P}$ in [LR] and [DT].

We now give more conditions equivalent to the existence of a linear extension morphism $s : E' \longrightarrow G'$ (and therefore equivalent to the existence of a linear extension morphism for polynomials). Other conditions may be found in [CGJ1] and in [D4, Prop. 6.18]. The implication ii) $\Rightarrow$ iii) is taken from [D3]. First, some definitions:

We say E is *finitely complemented* in G, if there is a constant c such that for all finite-dimensional subspaces S of G, there is a projector $\pi_S : S \longrightarrow E$ such that $\|\pi_S\| \leq c$, and $\pi_S(x) = x$ for $x \in S \cap E$.

If Y is a subspace of X we shall say, following Dineen [D3], that (Y, X) has the *polynomial extension property* if for all $k \in N$, each polynomial $P \in P(^kY)$ extends to a polynomial $\tilde{P} \in P(^kX)$. Note that it is not required that these extensions define a linear morphism. $(E)_{\mathcal{U}}$ denotes the ultrapower of E corresponding to the ultrafilter $\mathcal{U}$.

Theorem 2.9. *Let E be a subspace of G. Then the following are equivalent.*

i) There is a linear extension morphism $s : E' \longrightarrow G'$.
ii) $(E \times E', G \times E')$ has the polynomial extension property.
iii) E'' is complemented in G''.
iv) E is finitely complemented in G.
v) There is an ultrafilter $\mathcal{U}$ and a continuous linear function $T : G \longrightarrow (E)_{\mathcal{U}}$ such that $T(x) = x$ for $x \in E$.

Proof. i) implies ii): Consider $E' \times E''$ the dual of $E \times E'$ and $G' \times E''$ the dual of $G \times E'$ in the usual way. Then

$$E' \times E'' \longrightarrow G' \times E'' \text{ defined by } (\gamma, z) \mapsto (s(\gamma), z)$$

is a linear extension morphism from $(E \times E')'$ to $(G \times E')'$. Thus by the preceding theorem (with $F = C$) every k-homogeneous polynomial over $E \times E'$ extends to a k-homogeneous polynomial over $G \times E'$.

ii) implies iii): Define $P \in P^2(E \times E')$ by $P(x, \gamma) = \gamma(x)$. There is, by ii), a polynomial $\tilde{P} \in P^2(G \times E')$ which extends P. This polynomial extends by Aron-Berner to a polynomial $\overline{\tilde{P}}$ over $G'' \times E'''$. We consider Q its restriction to $G'' \times E'$. Now if $(b, \gamma) \in G'' \times E'$, let

$$\pi(b)(\gamma) = Q(b, \gamma).$$

Then $\pi(b)$ is an element of E'', and $\pi : G'' \longrightarrow E''$ is continuous and linear. Now if $z \in B_{E''}$ and $\gamma \in E'$, the weak*-closure of $B_E \times \{\gamma\}$ contains (z, γ). Using the Davie-Gamelin theorem, construct a net $(x_\alpha) \subset B_E$ weak*-converging to z and such that $Q(x_\alpha, \gamma) \to Q(z, \gamma)$. Then

$$\pi(z)(\gamma) = Q(z, \gamma) = \lim_\alpha Q(x_\alpha, \gamma) = \lim_\alpha \gamma(x_\alpha) = z(\gamma).$$

Hence $\pi(z) = z$, and π is a projection of G'' onto E''.

iii) implies iv): If $\pi : G'' \longrightarrow E''$ is a projector, take $c = \|\pi\|(1 + \varepsilon)$. For any finite-dimensional subspace S of G, $\pi(S)$ is a finite-dimensional subspace of E''. By the local reflexivity principle [De] there is a linear operator

$$T : \pi(S) \longrightarrow (T \circ \pi)(S) \subset E$$

such that $(1 - \varepsilon)\|x\| \leq \|T(x)\| \leq (1 + \varepsilon)\|x\|$ for $x \in \pi(S)$, and if $\pi(x) \in E$, $T(\pi(x)) = \pi(x)$. Then $\pi_S = T \circ \pi : S \longrightarrow \pi_S(S) \subset E$ has norm $\|\pi_S\| \leq \|T\|\|\pi\| \leq c$, and for $x \in S \cap E$ we have $\pi_S(x) = T(\pi(x)) = \pi(x) = x$.

iv) implies v): Consider the indexing set

$$I = \{S : S \text{ is a finite dimensional subspace of } G\}$$

ordered by inclusion, let $A_i = \{j : j \geq i\}$ and $\mathcal{U}$ be an ultrafilter on I containing the A_i's. For each $i = S \in I$ we have a projector $\pi_i : S \longrightarrow E$ of norm not exceeding c. Define for each $g \in G$

$$\widehat{\pi}_i(g) = \begin{cases} \pi_i(g) \text{ if } g \in S \\ 0 \text{ if } g \notin S \end{cases}$$

and let $T : G \longrightarrow (E)_{\mathcal{U}}$ be defined as $T(g) = [(\widehat{\pi}_i(g))]$ (the class of the family $(\widehat{\pi}_i(g))$ in the ultraproduct $(E)_{\mathcal{U}}$). T is linear –though the $\widehat{\pi}_i$'s are not– and $\|T(g)\| = \lim_{\mathcal{U}} \|\widehat{\pi}_i(g)\| \leq c\|g\|$. Now if $x \in E$, for all $i \geq S = \text{span}(x)$, $\pi_i(x) = x$, so $T(x) = [(\widehat{\pi}_i(x))] = [(x)]$.

v) implies i): First define $\sigma : E' \longrightarrow (E)'_{\mathcal{U}}$ as $\sigma(\gamma)([(x_i)]) = \lim_{\mathcal{U}} \gamma(x_i)$. Now set $s = T' \circ \sigma : E' \longrightarrow G'$. This is a linear extension morphism. Indeed, for all $\gamma \in E'$ and $x \in E$, we have

$$\begin{aligned} s(\gamma)(x) &= (T' \circ \sigma)(\gamma)(x) = (\sigma(\gamma) \circ T)(x) \\ &= \sigma(\gamma)[(x)] = \lim_{\mathcal{U}} \gamma(x) \\ &= \gamma(x). \end{aligned}$$

□

Some comments are in order. First, in iii) of the theorem we consider the bitranspose J'' of the inclusion $J : E \longrightarrow G$ to be the natural inclusion map of E'' in G''. This may, in some cases, be misleading. Consider for example the canonical inclusion $J_E : E \longrightarrow E''$. Then it is easily checked that the inclusion J''_E is not the canonical one $J_{E''} : E'' \longrightarrow E^{iv}$: indeed, let $\alpha \in E'''$ be non-zero, but such that $E \subset \text{Ker}\alpha$, and take $z \in E''$ such that $\alpha(z) = 1$. Then

$$J_{E''}(z)(\alpha) = \alpha(z) = 1 \neq 0 = z(J'_E(\alpha)) = J''_E(z)(\alpha).$$

The fact that $J_{E'''}$ and $J''_{E'}$ produce different extension morphisms

$$P(^kE'') \to P(^kE^{iv})$$

is related to the evaluation problem considered in [AGGM].

Note ([CGJ1], [Ka]) also that from the point of view of homological algebra, while complementation of E in G is the splitting of the sequence

$$0 \to E \to G \to G/E \to 0,$$

the properties in the Theorem are equivalent to the splitting of the dual sequence

$$0 \to (G/E)' \to G' \to E' \to 0.$$

Note also that one could write a 'norm sensitive' version of the theorem, putting 's norm preserving', $\|\pi\| = 1$, $c = 1 + \varepsilon$ for any ε, and $\|T\| = 1$.

We close this section with an application of the procedure $s \mapsto \overline{s}$ of Theorem 2.2. In [DD] Díaz and Dineen prove that if Banach spaces E, F have isomorphic duals, and E' has the Schur property and the approximation property, then $P(^kE)$ and $P(^kF)$ are isomorphic, i.e.: with the above hypotheses, the dual space E'

determines the space of polynomials $P(^kE)$. If $s : E' \to F'$ is an isomorphism, it seems natural to look into the map $\overline{s} : P(^kE) \to P(^kF)$. As mentioned after Theorem 2.2, in general $\overline{t \circ s} \neq \overline{t} \circ \overline{s}$. However, in the presence of Arens regularity, the procedure is sufficiently well-behaved to produce the following result (proved independently in [CCG] and [LZ]).

Theorem 2.10. *If E and F are symmetrically Arens-regular, and E' and F' are isomorphic (resp. isometric), then for any k, $P(^kE)$ and $P(^kF)$ are isomorphic (resp. isometric).* □

§3. Hahn-Banach type extensions

We now address the problem of when a continuous k-homogeneous scalar-valued polynomial P can be extended to any larger Banach space, regardless of whether other polynomials over the same space can be so extended and whether such extensions supply linear morphisms or not.

Following Kirwan and Ryan [KR], we shall say $P \in P(^kE)$ is *extendible* if it can be extended to a continuous polynomial over any Banach space G containing E as a subspace, and denote by $P_e(^kE)$ the space of all such polynomials. We start with some immediate comments on extendible polynomials.

Remark 3.1.

All extendible polynomials are weakly sequentially continuous. Indeed, consider E a subspace of $C(B_{E'})$ (the space of continuous functions over the closed unit ball of E' with the weak* topology), through the mapping $x \mapsto \widehat{x}$ where $\widehat{x}(\gamma) = \gamma(x)$. If P is an extendible polynomial over E, it extends to a polynomial $\widetilde{P}$ over $C(B_{E'})$. Since $C(B_{E'})$ has the Dunford-Pettis property, every polynomial defined on it is weakly sequentially continuous [Ry1] and this forces P to be weakly sequentially continuous as well.

Remark 3.2.

If $P \in P(^2E)$ is extendible, and $T : E \longrightarrow E'$ is the symmetric linear operator associated to P, then T is weakly compact: Call J the inclusion of E in $C(B_{E'})$ defined above, and let $\widetilde{P}$ be an extension of P. This means that $\widetilde{P} \circ J = P$, or equivalently, that there is a symmetric linear operator $S : C(B_{E'}) \longrightarrow C(B_{E'})'$ such that the following diagram commutes

$$\begin{tikzcd} E \arrow[r, "T"] \arrow[d, "J"'] & E' \\ C(B_{E'}) \arrow[r, "S"'] & C(B_{E'})' \arrow[u, "J'"'] \end{tikzcd}$$

Since $C(B_{E'})$ is symmetrically regular, S is weakly compact, and since J' is weak-to-weak continuous, $T(B_E) = J'(S(J(B_E)))$ is weakly precompact. Thus T is weakly compact.

Remark 3.3.

From [KR]: If H is a Hilbert space, and $P \in P(^2H)$, then P is extendible if and only if P is nuclear: Suppose first that P is extendible and, much as in the previous example, consider H contained in $\ell^\infty(B_{H'})$. On extending P one obtains the diagram

$$\begin{tikzcd} H \arrow[r, "T"] \arrow[d, "J"'] & H' \\ \ell^\infty(B_{H'}) \arrow[r, "S"'] & \ell^\infty(B_{H'})' \arrow[u, "J'"'] \end{tikzcd}$$

Since $\ell^\infty(B_{H'})'$ is an $\mathcal{L}_1$-space, S and J' are 2-summing, and so $J' \circ S$ is nuclear. Then $T = (J' \circ S) \circ J$ is nuclear, and so is P. The other implication is trivial, for all nuclear polynomials on any space are extendible.

We may now present a few concrete examples of non-extendible scalar-valued polynomials.

1) $P : \ell^2 \longrightarrow C$, given by $P(x) = \sum_k x_k^2$ cannot be extendible, for it is not weakly sequentially continuous: the unit basis in ℓ^2 tends weakly to 0, but $P(e_n) = 1$ for any n (also, P is not nuclear).

2) $P : \ell^1(Z) \longrightarrow C$, given by $P(x) = \sum_j \sum_k \operatorname{sgn}(j+k) x_j x_k$ is not extendible. This is the polynomial corresponding to the non-weakly compact operator given in Example 1.1, so it cannot be extendible.

3) $P : \ell^2 \longrightarrow C$, given by $P(x) = \sum_k \frac{x_k^2}{k}$ is not extendible, for it is not nuclear (and ℓ^2 is a Hilbert space). $P \in P_A(\ell^2)$; thus there are approximable polynomials which are not extendible.

There are, in general, many non-extendible scalar valued polynomials on a Banach space. There are some classes of polynomials, however, which are always extendible. All finite-type and all nuclear polynomials can be extended by simply applying the Hahn-Banach theorem to their component linear forms. It is also true that all integral polynomials over a Banach space are extendible [CZ1].

A problem related to the non-extendibility of polynomials and considered by P. Mazet in [Ma2] is the following: find the smallest constant c such that for any hyperplane F in G every 2-homogeneous polynomial P on F can be extended to G with the norm of the extension not exceeding $c\|P\|$. Clearly c must be larger than one, for otherwise using a transfinite induction argument every P would be extendible. Mazet finds the following bounds.

Theorem 3.4. *For real spaces, $c = 2$. For complex spaces, $\frac{7}{3} \leq c \leq 2\sqrt{2}$.* □

One way of extending polynomials defined on E to G would be to identify each with a linear form, and then apply the Hahn-Banach theorem to this linear form. Thus, we are interested in representing spaces of polynomials as dual spaces: say S' is algebraically isomorphic to a subspace P_E of $P(^kE)$, and that T' is algebraically isomorphic to a subspace P_G of $P(^kG)$, and that there is an linear

inyection $S \longrightarrow T$. Then if S is topologically a subspace of T, each $P \in P_E$ (considered as a linear form over S) may be extended by Hahn-Banach to $\widehat{P} \in P_G$. In fact, any given polynomial P will be extendible to $\widehat{P} \in P_G$ if and only if, as a linear functional on S it is continuous for the topology induced on S by T.

There have been several constructions of preduals of spaces of polynomials, and indeed, of spaces of holomorphic functions. In fact, the constructions usually produce more than just preduals. They produce solutions to the following problem: given a space of scalar-valued functions $\mathcal{F}(U)$ on the set U, can one construct a space $\mathcal{F}_*(U)$ and a function $e : U \longrightarrow \mathcal{F}_*(U)$ of the same *type* as those of $\mathcal{F}(U)$ factoring all functions of $\mathcal{F}(U)$ and identifying each with a continuous linear form $L_f : \mathcal{F}_*(U) \longrightarrow C$? That is,

$$\begin{array}{ccc} U & \xrightarrow{\ f\ } & C \\ {\scriptstyle e}\big\downarrow & & \nearrow_{L_f} \\ \mathcal{F}_*(U) & & \end{array}$$

One can then identify $\mathcal{F}(U)$ algebraically and topologically with the dual of $\mathcal{F}_*(U)$, and indeed, the space of F-valued functions $\mathcal{F}(U, F)$ with the continuous linear operators $L(\mathcal{F}_*(U), F)$. The existence of such a 'linearization' is strictly stronger than the mere existence of a predual.

Constructions of this type have been obtained for spaces of continuous homogeneous polynomials by R. Ryan [Ry2] through the use of symmetric tensor products. In the holomorphic setting, linearizations have been constructed for holomorphic functions by P. Mazet [Ma1] and J. Mujica and L. Nachbin [MN], for bounded holomorphic functions by J. Mujica [M2], and for holomorphic functions of bounded type by P. Galindo, D. García and M. Maestre [GGM1] and Mujica [M3] (see also [Boy]). A more abstract approach produces linearizations for a wide class of function spaces [CZ2].

We restrict our attention here to preduals of spaces of polynomials. Given E, let us denote by $\bigotimes_{s,k} E$ the vector space of symmetric k-fold tensor products of elements of E. Its algebraic dual $(\bigotimes_{s,k} E)^*$ is the space of all (not necesarily continuous) k-homogeneous polynomials over E. Furthermore, if $E \subset G$, there is a linear injection $\bigotimes_{s,k} E \longrightarrow \bigotimes_{s,k} G$. By requiring more or less stringent continuity conditions of the elements of $(\bigotimes_{s,k} E)^*$ one may obtain all classes of polynomials over E, in fact the following result holds [CZ1].

Proposition 3.5. *If Z is any subspace of $P(^kE)$ containing the finite type polynomials, Z is (algebraically) isomorphic to $(\bigotimes_{s,k} E, \tau)'$, where τ is a Hausdorff locally convex topology on $\bigotimes_{s,k} E$.* □

Note that C. Boyd and R. Ryan have proved that not all such subspaces of $P(^kE)$ are dual Banach spaces [BoyR]. We illustrate the proposition above by presenting a few of the classical polynomial spaces as duals of $\bigotimes_{s,k} E$ with the topologies indicated. We denote a typical element s of $\bigotimes_{s,k} E$ as a finite sum $s = \sum_j x_j^k$, where the x_j's are elements of E (this representation is not unique).

a) $P(^kE)$ is isometrically the dual of $\bigotimes_{s,k} E$ when this space is endowed with the projective tensor norm [Ry2], $\|s\| = \sup\{|\sum_j Q(x_j)| : Q \in P(^kE) \text{ with } \|Q\| \leq 1\}$.

b) $P_I(^kE)$ is isometrically the dual of $\bigotimes_{s,k} E$, when endowed with the following norm (equivalent to the injective tensor norm):

$$\|s\| = \sup\{|\sum_j \gamma(x_j)^k| : \gamma \in E' \text{ with } \|\gamma\| \leq 1\}.$$

c) $P_w(^kE)$ is algebraically the dual of $\bigotimes_{s,k} E$ when endowed with the following locally convex topology [CZ1]. Consider, for each compact subset K of E', the following seminorm on E: $\|x\|_K = \sup\{|\gamma(x)| : \gamma \in K\}$, and define the seminorm p_K on $\bigotimes_{s,k} E$ by $p_K(s) = \inf\{\sum_j \|x\|_K^k\}$, where the infimum ranges over all possible representations of s. If t is the topology induced by all such seminorms, we obtain an algebraic isomorphism $P_w(^kE) = (\bigotimes_{s,k} E, t)'$.

For more on tensor products of Banach spaces see [DF] and [Ry3].

We now review some results of Carando, Dimant, Kirwan, Pérez-García, Ryan, and Sevilla regarding the space $P_e(^kE)$ of all extendible polynomials over E. As a set, we have seen that $P_e(^kE)$ is contained in the space of weakly sequentially continuous polynomials, and that it contains the space of integral polynomials, but not in general the approximable polynomials. Kirwan and Ryan have proved the following.

Theorem 3.6. *Let P be an extendible polynomial. There is then a constant c such that for any Banach space G containing E, some extension of P to G has norm not larger than c.* □

In their proof they construct a Banach space which is an 'amalgamation' of all those which contain E, and extend P to this large space. It is therefore possible to define a norm on $P_e(^kE)$ by setting

$$\|P\|_e = \inf\{c > 0 : \text{ for all } G \text{ there is an extension of } P \text{ to } G \text{ with norm } \leq c\}$$

This norm is larger than the usual norm in $P(^kE)$, and is equivalent to it if and only if every polynomial is extendible. Also, $(P_e(^kE), \|\cdot\|_e)$ is complete, and a predual of it is obtained by putting in $\bigotimes_{s,k} E$ the norm

$$\|s\|_\eta = \inf\{\sum_j \|g_j\|^k : s = \sum_j g_j^k, g_j \in G\},$$

where the G's range over all Banach spaces containing E as a subspace.

In [C1], Carando takes a different approach to obtain a predual of $P_e(^kE)$. Starting with the inclusion $J : E \longrightarrow C(B_{E'})$ defined in Remark 3.1, he considers

$$\bigotimes_{s,k} J : \bigotimes_{s,k} E \longrightarrow \bigotimes_{s,k} C(B_{E'}),$$

putting in $\bigotimes_{s,k} C(B_{E'})$ the projective norm, and then considering in $\bigotimes_{s,k} E$ the norm induced by this map.

Apart from the map $J : E \to C(B_{E'})$, another inclusion which has proved useful in the study of extendible polynomials is

$$E \to \ell^\infty(B_{E'}) \text{ given by } x \mapsto (\gamma(x))_{\gamma \in B_{E'}}.$$

$\ell^\infty(B_{E'})$ has the metric extension property [DF], that is, given a linear map $T : X \to \ell^\infty(B_{E'})$ and $Y \supseteqq X$, T admits a norm-preserving extension to Y. The following results are adapted from [C1] and [CGJ1].

Theorem 3.7. *Let P be a scalar-valued k-homogeneous polynomial over E. Then the following are equivalent:*
a) P is extendible.
b) P extends to $C(B_{E'})$.
b') For some compact set K, there is a linear operator $S : E \to C(K)$, and $Q \in P^k(C(K))$ such that $P = Q \circ S$.
c) P extends to $\ell^\infty(B_{E'})$.
c') For some set I, there is a linear operator $S : E \to \ell^\infty(I)$, and $Q \in P^k(\ell^\infty(I))$ such that $P = Q \circ S$. □

In [CGJ1] the authors study extendibility of bilinear forms, and obtain, among other results:

Theorem 3.8. *Let $A \in L^2(E)$. Then the following are equivalent.*
a) A is extedible.
b) There exists an $\mathcal{L}_\infty$-space and operators $u, v : E \to \mathcal{L}_\infty$ and $B \in L^2(\mathcal{L}_\infty)$ such that $A(x, y) = B(u(x), v(y))$.
c) There exists a Hilbert space H and two 2-summing operators $u, v : E \to H$ such that $A(x, y) =< u(x), v(y) >$. □

Results similar to those of Theorem 3.6 for vector-valued polynomials can be found in [C1]. There the following results for a k-homogeneous polynomial $P : E \longrightarrow F$ are obtained.

Theorem 3.9. *P is extendible if and only if P extends to $\ell^\infty(B_{E'})$. If F is complemented in its bidual, P is extendible if and only if P extends to $C(B_{E'})$.* □

An immediate corollary of b') above is

Corollary 3.10. *If $P \in P(^kE)$ is extendible, and $T : X \to E$ linear, then $P \circ T$ is extendible and $\|P \circ T\|_e \leq \|P\|_e \|T\|^k$.* □

Thus, any restriction of an extendible polynomial is extendible. Also, if P is extendible, so is its Aron-Berner extension. By c), and the fact that $\ell^\infty(B_{E'})$ has the metric extension property, the norm of any extension of P to any space can be bounded by the norm of any extension of P to $\ell^\infty(B_{E'})$. Thus the 'extendible' norm $\|.\|_e$ defined above can also be expressed as

$$\|P\|_e = \inf\{\|Q\| : Q \in P(^k\ell^\infty(B_{E'})) \text{ and } Q \text{ extends } P\}.$$

In [P] Pisier gave a counterexample to a well-known conjecture of Grothendieck [G] regarding the non-existence of non-nuclear locally convex spaces E and F such that the injective and projective tensor norms coincide on the tensor product $E \bigotimes F$. Pisier constructed a Banach space P such that $P \bigotimes_\varepsilon P \simeq P \bigotimes_\pi P$. Thus, over Pisier's space all 2-homogeneous polynomials are integral. The situation is very different for polynomials of degree higher than 2. Indeed, [Pg] has shown that any infinite-dimensional Banach space admits extendible non-integral polynomials of any degree higher that 3, and in [CD] Carando and Dimant close the gap by proving the same for any degree higher than 2. Their construction makes use of finite-dimensional estimates of Boas [Boa].

Any space of cotype 2 (this includes ℓ^p spaces for $1 \leq p \leq 2$) can be embedded in Pisier's space [P]. Thus if E has cotype 2, all extendible 2-homogenesous polynomials over E are integral. Indeed, an extension to Pisier's space will be integral, and thus also the original polynomial ([CGJ1], [C2]).

Regarding extendible polynomials on ℓ^p spaces, Carando, Dimant and Sevilla [C2] [CDS] have constructed on all ℓ^p spaces with $p > 2$ extendible polynomials which are not integral.

Polynomials of degree two and bilinear forms have attracted particular attention in recent years. In [CGJ2] a homological approach is applied to the problem of extendibility of bilinear forms from E to G. Considerations of pushout and pullback diagrams produce the results:

Proposition 3.11. *If for all X the sequences*

$$0 \to (G/E)' \to X \to E \to 0 \qquad \textit{and} \qquad 0 \to G' \to X \to G/E \to 0$$

split, then all bilinear forms on E extend to G. If for all X the sequences

$$0 \to E' \to X \to G/E \to 0 \qquad \textit{and} \qquad 0 \to (G/E)' \to X \to G \to 0$$

split, then all bilinear forms on E extend to G. □

Since some of these properties are known when G is an $\mathcal{L}_1$-space, the authors concentrate on these and produce the following:

Theorem 3.12. *If G is an $\mathcal{L}_1$-space and $E \subset G$, the following are equivalent.*

a) All bilinear forms on E extend to G.

b) All exact sequences $0 \to E' \to X \to G/E \to 0$ split.

c) All exact sequences $0 \to (G/E)' \to X \to E \to 0$ split. □

References

[A] Alencar, R., *Multilinear mappings of nuclear and integral type*, Proc. Amer. Math. Soc. **94 (1)** (1985), 33-38.

[A1] Arens, R., *The adjoint of a bilinear operation*, Proc. Amer. Math. Soc. **2** (1951), 839-848.

[A2] Arens, R., *Operations induced in function classes*, Monatsh. Math. **55** (1951), 1-19.

[AB] Aron, R. and Berner, P., *A Hahn-Banach extension theorem for analytic mappings*, Bull. Soc. Math. France **106 (1)** (1978), 3-24.

[ABC] Aron, R., Boyd, C. and Choi, Y. S., *Unique Hahn-Banach theorems for spaces of homogeneous polynomials*, J. Aust. Math. Soc. **70 (3)** (2001), 387-400.

[ACG1] Aron, R., Cole, B., and Gamelin, T., *Spectra of algebras of analytic functions on a Banach space*, J. Reine Angew. Math. **415** (1991), 51-93.

[ACG2] Aron, R., Cole, B., and Gamelin, T., *Weak-star continuous analytic functions*, Canad. Jour. Math. **47 (4)** (1995), 673-683.

[AD] Aron, R. and Dineen, S., *Q-reflexive Banach spaces*, Rocky Mountain Jour. Math. **27(4)** (1997), 1009-1025.

[AG] Aron, R. and Galindo, P., *Weakly compact multilinear mappings*, Proc. Edinburgh Math. Soc. **40** (1997), 181-192.

[AGGM] Aron, R., Galindo, P., García, D., and Maestre, M., *Regularity and algebras of analytic functions in infinite dimensions*, Trans. Amer. Math. Soc. **348 (2)** (1996), 543-559.

[AHV] Aron, R., Hervés, C. and Valdivia, M., *Weakly continuous mappings on Banach spaces*, Jour. Funct. Anal. **52 (2)** (1983), 189-204.

[ALRR] Aron, R., Lindström, M., Ruess, W. and Ryan, R., *Uniform factorization for compact sets of operators*, Proc. Amer. Math. Soc. **127(4)** (1999), 1119-1125.

[AP] Aron, R. and Prolla, J., *Polynomial approximation of differentiable functions on Banach spaces*, Jour. Reine Angew. Math. **313** (1980), 195-216.

[AS] Aron, R. and Schottenloher, M., *Compact holomorphic mappings on Banach spaces and the approximation property*, Jour. Funct. Anal. **21 (1)** (1976), 7-30.

[Ba] Barroso, J., *Introduction to holomorphy* North-Holland Math. Studies, 106, 1985.

[BLLl] Benyamini, Y., Lassalle, S. and Llavona, J. L. G., *Homogeneous orthogonally-additive polynomials on Banach lattices*, Bull. London Math. Soc. **38(3)** (2006), 459-469.

[Boa] Boas, H. P., *Majorant series*, J. Korean Math. Soc. **37(2)** (2000), 321-337.

[Bol] Boland, P., *Holomorphic functions on nuclear spaces*, Trans. Amer. Math. Soc. **209** (1975), 275-281.

[Boy] Boyd, C., *Montel and reflexive preduals of spaces of holomorphic functions on Fréchet spaces*, Studia Math. **107(3)** (1993), 305-315.

[BoyR] Boyd, C. and Ryan, R., *Geometric theory of spaces of integral polynomials and symmetric tensor products*, J. Funct. Anal. **179(1)** (2001), 18-42.

[CCG] Cabello Sánchez, F., Castillo, J. M. F. and García, R., *Polynomials on dual-isomorphic spaces* Ark. Mat. **38** (2000), 37-44.

[C1] Carando, D., *Extendible polynomials on Banach spaces*, J. Math. Anal. Appl. **233(1)** (1999), 359-372.

[C2] Carando, D., *Extendibility of polynomials and analytic functions on ℓ^p*, Studia Math. **145(1)** (2001), 67-73.

[CD] Carando, D. and Dimant, V., *Extension of polynomials and John's theorem for symmetric tensor products* to appear in Proc. of the Amer. Math. Soc.

[CDDL] Carando, D., Dimant, V., Duarte, B. and Lassalle, S., *K-bounded polynomials*, Math. Proc. Royal Irish Acad. **98A(2)** (1998), 159-171.

[CDS] Carando, D., Dimant, V. and Sevilla, P., *Limit orders and multilinear forms on ℓ^p spaces*, Publ. RIMS **42** (2006), 507-522.

[CLZ] Carando, D., Lassalle, S. and Zalduendo, I., *Orthogonally additive polynomials over $C(K)$ are measures –a short proof* to appear in Int. Eq. and Oper. Theory.

[CZ1] Carando, D. and Zalduendo, I., *A Hahn-Banach theorem for integral polynomials*, Proc. Amer. Math. Soc. **127** (1999), 241-250.

[CZ2] Carando, D. and Zalduendo, I., *Linearization of functions*, Math. Ann. **328** (2004), 683-700.

[CGJ1] Castillo, J. M. F., García, R. and Jaramillo, J. A., *Extension of bilinear forms on Banach spaces*, Proc. Amer. Math. Soc. **129(12)** (2001), 3647-3656.

[CGJ2] Castillo, J. M. F., García, R. and Jaramillo, J. A., *Extension of bilinear forms from subspaces of $\mathcal{L}_\infty$-spaces*, Ann. Acad. Sci. Fenn. **27** (2002), 91-96.

[CM] Colombeau, J. F. and Mujica, J., *Holomorphic and differentiable mappings of uniform bounded type*, North-Holland Math. Studies **71** (1982), 179-200.

[DG] Davie, A. and Gamelin, T., *A theorem on polynomial-star approximation* Proc. Amer. Math. Soc. **106 (2)** (1989), 351-356.

[De] Dean, D., *The equation $L(E, X^{**}) = L(E, X)^{**}$ and the principle of local reflexivity*, Proc. Amer Math. Soc. **40** (1973), 146-148.

[DF] Defant, A. and Floret, K., *Tensor norms and operator ideals*, North-Holland, Amsterdam, 1993.

[DD] Díaz, J. C. and Dineen, S., *Polynomials on stable spaces*, Ark. Mat. **36** (1998) 87-96.

[D1] Dineen, S., *Holomorphically complete locally convex topological vector spaces*, LNM **332** (1973), 77-111.

[D2] Dineen, S., *Complex analysis in locally convex spaces*, North-Holland Math. Studies, 57, 1981.

[D3] Dineen, S., *Quasinormable spaces of holomorphic functions* Note Mat. **13 (1)** (1993), 155-195.

[D4] Dineen, S., *Complex analysis on infinite dimensional spaces*, Springer-Verlag, London, 1999.

[DT] Dineen, S. and Timoney, R., *Complex geodesics on convex domains*, North-Holland Math. Studies **170** (1992), 333-365.

[DH] Duncan, J. and Hosseiniun, S., *The second dual of a Banach algebra* Proc. Royal Soc. Edinb. **A84** (1979), 309-325.

[FGLl] Ferrera, J., Gómez, J. and Llavona, J., *On completion of spaces of weakly continuous functions*, Bull. London Math. Soc. **15** (1983), 260-264.

[GGM1] Galindo, P., García, D. and Maestre, M., *Holomorphic mappings of bounded type*, J. Math. Anal. Appl. **166(1)** (1992), 236-246.

[GGM2] Galindo, P., García, D. and Maestre, M., *Entire functions of bounded type on Fréchet spaces*, Math. Nachr. **161** (1993), 185-198.

[GGMM] Galindo, P., García, D., Maestre, M., and Mujica, J., *Extensions of multilinear mappings on Banach spaces*, Studia Math. **108 (1)** (1994), 55-76.

[GI] Godefroy, G. and Iochum, B., *Arens-regularity of Banach algebras and geometry of Banach spaces*, Jour. Funct. Anal. **80** (1988), 47-59.

[GG] González, M. and Gutiérrez, J., *When every polynomial is unconditionally converging*, Arch. Math. **63 (2)** (1994), 145-151.

[G] Grothendieck, A. *Résumé de la théorie métrique des produits tensoriels topologiques*, Bol. Soc. Mat. Sao Paulo **8** (1956), 83-110.

[GJLl] Gutiérrez, J., Jaramillo, J. and Llavona, J., *Polynomials and geometry of Banach spaces*, Extracta Math. **10 (2)** (1995), 79-114.

[H] Haydon, R., *Some more characterizations of Banach spaces containing* ℓ^1, Math. Proc. Cambridge Phil. Soc. **80 (2)** (1976), 269-276.

[HP] Hille, E. and Phillips, R., *Functional analysis and semigroups*, Amer. Math. Soc., Providence, RI, 1974.

[IL] Iochum, B. and Loupias, G., *Arens regularity and local reflexivity principle for Banach algebras*, Math. Ann. **284 (1)** (1989), 23-40.

[JPZ] Jaramillo, J., Prieto, A. and Zalduendo, I., *The bidual of the space of polynomials on a Banach space*, Math. Proc. Cambridge Phil. Soc. **122** (1997), 457-471.

[Ka] Kalton, N., *Locally complemented subspaces and* $\mathcal{L}_p$*-spaces for* $0 < p < 1$, Math. Nachr. **115** (1984), 71-97.

[Kh1] Khue, N. V., *On the extension of holomorphic maps on locally convex space*, Ann. Polon. Math. **43(3)** (1983), 267-282.

[Kh2] Khue, N. V., *On the extension of holomorphic functions on locally conves spaces with values in Fréchet spaces*, Ann. Polon. Math. **44(2)** (1984), 163-175.

[KR] Kirwan, P. and Ryan, R., *Extendibility of homogeneous polynomials on Banach spaces*, Proc. Amer. Math. Soc. **126(4)** (1998), 1023-1029.

[LZ] Lassalle, S. and Zalduendo, I., *To what extent does the dual Banach space* E' *determine the polynomials over* E*?*, Ark. Mat. **38** (2000), 343-354.

[LR] Lindström, M. and Ryan, R., *Applications of ultraproducts to infinite-dimensional holomorphy*, Math. Scand. **71 (2)** (1992), 229-242.

[Ma1] Mazet, P., *Analytic sets in locally convex spaces*, North-Holland Mathematics Studies **89** (1984).

[Ma2] Mazet, P., *A Hahn-Banach theorem for quadratic forms*, Math. Proc. of the Royal Irish Ac. **100A(2)** (2000), 177-182.

[MV] Meise, R. and Vogt, D., *Extension of entire functions on nuclear locally convex spaces*, Proc. Amer. Math. Soc. **92 (4)** (1984), 494-500.

[Mo1] Moraes, L., *The Hahn-Banach extension theorem for some spaces of n-homogeneous polynomials*, North-Holland Math. Studies **90** (1984), 265-274.

[Mo2] Moraes, L., *A Hahn-Banach extension theorem for some holomorphic functions*, North-Holland Math. Studies **125** (1986), 205-220.

[Mo3] Moraes, L., *Extension of holomorphic mappings from E to E''*, Proc. Amer. Math. Soc. **118 (2)** (1993), 455-461.

[M1] Mujica, J., *Complex analysis in Banach spaces*, North-Holland Math. Studies, 120, Amsterdam, 1986.

[M2] Mujica, J., *Linearization of bounded holomorphic mappings on Banach spaces*, Trans. Amer. Math. Soc. **324** (1991), 867-887.

[M3] Mujica, J., *Linearization of holomorphic mappings of bounded type*, Progress in Functional Analysis (1992), North-Holland, Amsterdam, 149-162.

[MN] Mujica, J. and Nachbin, L., *Linearization of holomorphic mappings on locally convex spaces*, J. Math. Pures Appl. **71** (1992), 543-560.

[N] Nachbin, L., *Topology on spaces of holomorphic mappings*, Springer-Verlag, New York, 1969.

[Ni] Nicodemi, O., *Homomorphisms of algebras of germs of holomorphic functions*, LNM **843** (1981), 534-546.

[Pg] Pérez-García, D., *A counterexample using 4-linear forms*, Bull. Aust. Math. Soc. **70(3)** (2004), 469-473.

[PgV] Pérez-García, D. and Villanueva, I., *Orthogonally additive polynomials on spaces of continuous functions*, J. Math. Anal. Appl. **306(1)** (2005), 97-105.

[PZ] Pinasco, D. and Zalduendo, I., *Integral representation of holomorphic functions on Banach spaces*, J. Math. Anal. Appl. **308** (2005), 159-174.

[P] Pisier, G., *Counterexamples to a conjecture of Grothendieck*, Acta Math. **151** (1983), 181-208.

[Re] Rennison, J., *A note on the extension of associative products*, Proc. Amer. Math. Soc. **17 (6)** (1966), 1375-1377.

[Rp] Rodríguez Palacios, A., *A note on Arens regularity*, Quart. Jour. Math. Oxford Ser. 2 **38 (149)** (1987), 91-93.

[Ry1] Ryan, R., *Dunford-Pettis properties*, Bull. Acad. Polon. Sci. **27** (1979), 373-379.

[Ry2] Ryan, R., *Applications of topological tensor products to infinite dimensional holomorphy*, Ph. D. thesis, University College, Dublin, 1980.

[Ry3] Ryan, R., *Introduction to tensor products of Banach spaces*, Springer-Verlag, London, 2002.

[S] Stegall, C., *Duals of certain spaces with the Dunford-Pettis property*, Notices Amer. Math. Soc. **19 (7)** (1972), A799.

[T] Toma, E., *Aplicaçoes holomorfas e polinomios τ-continuos*, Ph. D. thesis, Universidade Federal do Rio de Janeiro, Rio de Janeiro, 1993.

[Z1] Zalduendo, I., *A canonical extension for analytic functions on Banach spaces*, Trans. Amer. Math. Soc. **320 (2)** (1990), 747-763.

[Z2] Zalduendo, I., *A simple example of a non-commutative Arens product*, Publ. Mat. **35 (2)** (1991), 475-477.

[Z3] Zalduendo, I., *Extending polynomials–a survey*, Publicaciones del Departamento de Análisis Matemático - Univerisdad Complutense de Madrid **secc. 1, Num. 41** (1998).

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Recibido: 12 de setiembre de 2005
Aceptado: 7 de agosto de 2006