

SPECIAL METRICS IN G_2 GEOMETRY

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ABSTRACT. We discuss metrics with holonomy G_2 by presenting a few crucial examples and review a series of G_2 manifolds constructed via solvable Lie groups, obtained in [15]. These carry two related distinguished metrics, one negative Einstein and the other in the conformal class of a Ricci-flat metric, plus other features considered definitely worth investigating.

1. INTRODUCTION

A seven-dimensional Riemannian manifold is called a G_2 -manifold whenever the structure group of the tangent bundle is contained in the subgroup

$$G_2 \subset SO(7)$$

of the orthogonal group. Admitting such a reduction is equivalent to the existence of a non-degenerate three-form of positive type φ . When this form is covariantly constant with respect to the Levi-Civita connection then the holonomy of the manifold is contained in G_2 , and the corresponding manifold is called parallel. Despite major advances in understanding exceptional geometry, producing metrics with holonomy equal to G_2 is still today not that easy. A quick browse through the literature of high-energy physics gives evidence of this, although a rather good assortment is available. Exhibiting *complete* metrics, instead, has remained arduous since the first examples were constructed by Bryant and Salamon [12] almost twenty years ago: these are built from the inclusion $SO(4) \subset G_2$ on vector bundles over 3- and 4-manifolds, and have been relentlessly referred to ever since, hence becoming somehow ‘classical’. We recall one construction in Section 3, by which the virtues of that landmark paper will be even more apparent.

This note consists of two parts, the first of which pretentiously tries to collect material on G_2 structures placing emphasis on the Riemannian aspects. Needless to say, the exposition is incomplete, hence we recommend to begin with [10], [32] and the survey [33]. The second half of this paper reviews the results obtained in previous work [15] and describes in some detail two solutions of the Hitchin flow given by metrics with holonomy equal to the full G_2 .

Two incomplete holonomy metrics with a 2-step nilpotent isometry group N , whose orbits are hypersurfaces realised as torus bundles over tori, were presented in [21]. It was shown that such Ricci-flat metrics are intimately related to complete

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Einstein manifolds with a transitive solvable group of isometries. The metrics arise from Heisenberg limits of the isometry group of the two complete cohomogeneity-one metrics of [12], once again leading back to these examples. They are moreover scale-invariant, that is to say they have additional symmetries generated by a conformal Killing vector.

In [15] we showed that the previous Ricci-flat metrics are conformal to complete homogeneous metrics on a special kind of solvable Lie group. This is a rank-one solvable extension of N equipped with an $SU(3)$ -structure (J, ω, ψ^+) , where (J, ω) is an almost Hermitian structure and ψ^+ a 3-form of unit norm. The extension is determined by a derivation D of the metric Lie algebra $(\mathfrak{n}, \langle, \rangle)$ of N . Such a derivation is non-singular, self-adjoint with respect to \langle, \rangle and satisfies $(DJ)^2 = (JD)^2$, a relation of primary importance for the construction. Conformally parallel G_2 structures on rank-one solvable extensions of 6-dimensional nilpotent Lie algebras \mathfrak{n} endowed with an $SU(3)$ structure and a derivation D satisfying the previous requirements were then studied. We described thoroughly the corresponding metrics with holonomy contained in G_2 obtained after the conformal change. These fall into two categories: one where the holonomy is properly contained in the exceptional group, and the other consisting of structures with holonomy truly equal to G_2 , depending upon whether the Lie algebra is irreducible. This remark is often useful if one intends to avoid the sometimes daunting task of detecting special spinor fields, as carried out in [4]. The latter uncovers the peculiar spinorial behaviour of these manifolds and might shed light on some applications to the physics of high energies. As a consequence of the fact that g belongs to the same conformal class of the holonomy metric, the $SU(3)$ structure on the Lie group N is half-flat [16]. A breakthrough result proved by Hitchin [23] predicts then that there is a (usually incomplete) metric \tilde{g} with $Hol(\tilde{g}) \subseteq G_2$ on the product of N with some real interval, but this is hard to determine explicitly. Nevertheless, due to the convenient set-up, we computed the solution of the evolution equations for each of the half-flat $SU(3)$ structures. See Section 7 where exhaustive computations are carried out for the structures relative to the nilpotent Lie groups isomorphic to

$$\begin{aligned} [e_2, e_1] &= e_4, & [e_3, e_1] &= e_5, & [e_3, e_2] &= e_6 & \text{and} \\ [e_2, e_1] &= e_5, & [e_4, e_1] &= [e_3, e_2] &= e_6. \end{aligned}$$

Comparing the solutions found in this way with the metric obtained with the conformal change we proved that the metrics, though arising by two completely different methods, coincide, as one should rightly expect.

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2. $SU(3)$ AND G_2 STRUCTURES

Let N^6 be a real six-dimensional manifold. An $SU(3)$ -reduction is given by an almost Hermitian triple (J, h, ω) consisting of a Riemannian metric h , an h -orthogonal complex structure J , the induced $(1, 1)$ -form ω , together with a $(3, 0)$ -form of unit norm ψ . Let

$$\psi^+ = \frac{1}{2}(\psi + \bar{\psi}), \quad \psi^- = \frac{1}{2i}(\psi - \bar{\psi})$$

be the real and imaginary parts of ψ . There is an orthonormal basis of 1-forms such that

$$\begin{aligned} \omega &= e^{14} - e^{23} + e^{56}, \\ \psi &= (e^1 + ie^4) \wedge (e^2 - ie^3) \wedge (e^5 + ie^6). \end{aligned} \tag{2.1}$$

In the paper we will always indicate by (e^i) a – possibly local – basis of 1-forms, and set $e^{ij} = e^i \wedge e^j$. Dual vectors will be denoted by lower indexes, so $e^k(e_l) = \delta_{kl}$. Observe that the differential forms defining the reduction satisfy $\psi \wedge \omega = 0$ and $3\psi \wedge \bar{\psi} = 4i\omega^3$. Since ψ^+ is chosen to have stabiliser $SL(3, \mathbb{C})$ in the general linear group, it determines the almost complex structure J and $\psi^- = J\psi^+$ [23]. With $\mathfrak{su}(3)^\perp = \mathfrak{so}(6)/\mathfrak{su}(3)$ denoting the orthogonal complement of $\mathfrak{su}(3)$ in $\mathfrak{so}(6)$, the known identifications

$$[\Lambda_0^{1,1}] \cong \mathfrak{su}(3), \quad \mathbb{R} \oplus [\Lambda^{2,0}] \cong \mathfrak{su}(3)^\perp$$

allow one to split the space $T^*N \otimes \mathfrak{su}(3)^\perp = \mathcal{W}$ in 7 irreducible $SU(3)$ -submodules

$$\mathcal{W} = \mathcal{W}_1^+ \oplus \mathcal{W}_1^- \oplus \mathcal{W}_2^+ \oplus \mathcal{W}_2^- \oplus \mathcal{W}_3 \oplus \mathcal{W}_4 \oplus \mathcal{W}_5.$$

The intrinsic torsion, a tensor $\tau \in \mathcal{W}$, accounts for the ‘non-integrable’ structures, for the holonomy of the Riemannian metric h is contained in $SU(3)$ if and only if ω, ψ^+ and ψ^- are all closed, in other words when $\tau = 0$. The $SU(3)$ -representations \mathcal{W}_i implement the original Gray–Hervella classes, and the reader might be familiar with some names, like

- nearly Kähler structures, for which $\tau \in \mathcal{W}_1^+ \oplus \mathcal{W}_1^-$;
- symplectic structures, where $\tau \in \mathcal{W}_2^+ \oplus \mathcal{W}_2^- \oplus \mathcal{W}_5$;
- Hermitian structures, corresponding to $\tau \in \mathcal{W}_3 \oplus \mathcal{W}_4$,

but will probably not be acquainted with the remarkable

- half-flat class $\mathcal{W}_1^- \oplus \mathcal{W}_2^- \oplus \mathcal{W}_3$.

It is easily seen that picking τ in this latter space is the same as demanding that ψ^+ and $\omega^2 = \omega \wedge \omega$ be closed forms. The name is designed to remind the fact that of the original dimension of \mathcal{W} , only half survives.

We say that a 7-dimensional manifold M^7 is *built* from N if the cotangent space of M splits at each point $m \in M$ as

$$T_m^*M^7 = T_n^*N^6 \oplus \mathbb{R}.$$

Hypersurfaces $N \subset M$, fibre bundles $M \rightarrow N$, or quotients M/S^1 are instances thereof. As $SU(3)$ is a maximal subgroup of G_2 , the special Hermitian geometry of N induces a differential form on M^7 (pullbacks omitted)

$$\varphi = \omega \wedge e^7 + \psi^+, \tag{2.2}$$

where e^7 is a 1-form on \mathbb{R} . The three-form φ has isotropy $G_2 \subset SO(7)$ and determines a compatible Riemannian metric g and the 4-form $*\varphi = \psi^- \wedge e^7 + \frac{1}{2}\omega^2$, via the Hodge operator $*$. Completing the basis of T^*N with e^7 preserves orthonormality and

$$\varphi = e^{147} - e^{237} + e^{567} + e^{125} + e^{136} + e^{246} - e^{345}. \quad (2.3)$$

If (and only if) M is parallel, φ and $*\varphi$ become closed [19] and the induced metric g has zero Ricci curvature [9].

The intrinsic torsion of a G_2 structure can be identified with the covariant derivative of the fundamental form with respect to the Levi-Civita connection ∇ . In [19] (see also [14]) a classification of G_2 -manifolds in 16 classes is given by studying the G_2 -irreducible components of the torsion space \mathcal{T} . Fernández and Gray proved that \mathcal{T} consists of tensors having the same symmetries as $\nabla\varphi$ and consists of four G_2 -irreducible components \mathcal{T}_i , $i = 0, \dots, 3$. On a G_2 -manifold, the group's action on the tangent spaces $T_m M \cong \mathbb{R}^7 = V$ induces an action on the exterior algebra $\Lambda^p(M)$. There are decompositions into modules

$$\begin{aligned} \Lambda^2 V^* &= \Lambda_{14}^2 \oplus \Lambda_7^2 \cong \Lambda^5 V^*, \\ \Lambda^3 V^* &= \Lambda_{27}^3 \oplus \Lambda_7^3 \oplus \Lambda_1^3 \cong \Lambda^4 V^*, \end{aligned}$$

where Λ_p^k denotes a certain irreducible G_2 -module of dimension p . The intrinsic torsion of the G_2 -structure is encoded in the exterior derivatives $d\varphi, d*\varphi$ as follows

$$\begin{aligned} d\varphi &= \tau_0 * \varphi + 3\tau_1 \wedge \varphi + * \tau_3, \\ d*\varphi &= 4\tau_1 \wedge * \varphi + \tau_2 \wedge \varphi, \end{aligned}$$

for unique differential forms

$$\begin{aligned} \tau_0 &\in \mathbb{R} \cong \mathcal{T}_0, \\ \tau_1 &\in \Lambda^1 \cong \mathcal{T}_1, \\ \tau_2 &\in \Lambda_{14}^2 \cong \mathfrak{g}_2 \text{ (the exceptional Lie algebra)} \cong \mathcal{T}_2, \\ \tau_3 &\in \Lambda_{27}^3 \cong \mathcal{S}_0^2 V^* \text{ (the space of traceless symmetric 2-tensors)} \cong \mathcal{T}_3, \end{aligned}$$

see for instance [13, 11].

Friedrich and Ivanov proved that $\tau_2 = 0$ if and only if there exists an affine connection $\tilde{\nabla}$ with totally skew-symmetric torsion T such that $\tilde{\nabla}\varphi = 0$ [20]. Then M^7 is a ' G_2 -manifold with torsion' (G_2 T) and the resulting torsion 3-tensor is

$$T = \frac{7}{6}\tau_0\varphi - *d\varphi + *(4\tau_1 \wedge \varphi).$$

An interesting subset of G_2 T-manifolds consists of those of type \mathcal{T}_1 , for which

$$\begin{aligned} d\varphi &= 3\tau_1 \wedge \varphi, \\ d(*\varphi) &= 4\tau_1 \wedge * \varphi \end{aligned}$$

so

$$T = *(\tau_1 \wedge \varphi).$$

These manifolds are also called locally conformally parallel, since the change $e^{2f}g$ (with $df = -\frac{1}{3}\tau_1$) gives locally a parallel structure.

The reader interested in *compact* G_2 manifolds of class \mathcal{T}_1 should look at [24].

3. SOME EXAMPLES OF G_2 METRICS

1. We recall one essential idea of [31]. Let K be an oriented Riemannian 4-manifold with local orthonormal basis f^4, f^5, f^6, f^7 for the cotangent bundle. Define the unit forms

$$f^1 = f^4 f^5 - f^6 f^7, \quad f^2 = f^4 f^6 - f^7 f^5, \quad f^3 = f^4 f^7 - f^5 f^6 \tag{3.1}$$

to span $\Lambda^2 T^*K$. The total space Y of the latter decomposes as $H \oplus V$, where the vertical space V is generated by three 1-forms (e^j) on Y depending on the fibre coordinates, whilst H has basis (the pullbacks of) (3.1). Given now two positive functions α, β on Y ,

$$\phi = 6\alpha^3 e^1 e^2 e^3 + \alpha\beta^2 d(f^1 e^1 + f^2 e^2 + f^3 e^3)$$

is a G_2 structure determining a Riemannian metric of the form $\alpha^2 g_V + \beta^2 g_H$ in terms of the above splitting. Now if K is self-dual and positive-Einstein, choosing $\beta = (tr + 1)^{1/4}$, $\alpha = \beta^{-1}$, with $r > 0$ a radial coordinate and some positive t , renders φ closed and coclosed, hence parallel, and the metric

$$(tr + 1)^{-1/2} g_V + (tr + 1)^{1/2} g_H \tag{3.2}$$

is complete, Ricci-flat and has holonomy equal to G_2 . When the parameter tends to zero, the metric becomes conical on the product of \mathbb{R}^+ with the twistor space of K . Since K is S^4 or $\mathbb{C}\mathbb{P}^2$, the groups $SO(5)$, $SU(3)$ act isometrically with generic orbits of codimension one. The metric resembles the Eguchi-Hanson instanton [18], which is Einstein on $T^*\mathbb{C}\mathbb{P}^1$ and makes the standard holomorphic symplectic form covariantly constant.

2. A similar example, constructed in the flavour of Section 2, is the following. Let Iw be the compact quotient of the complex 3-dimensional Heisenberg group by the Gaussian integers, called the Iwasawa manifold. The product $Iw \times \mathbb{R}$ admits an orthonormal basis (e^j) with $e^j = f^j$ of (3.1) for $j \geq 4$, such that

$$de^j = \begin{cases} f^j & j = 1, 2, 3 \\ 0 & j = 4, 5, 6, 7 \end{cases}$$

and the three-form

$$\phi = e^{127} + e^{347} + e^{657} - e^{135} + e^{126} + e^{643} + e^{254}$$

is a G_2 structure. Indicating by \lrcorner the interior product of a vector with a differential form, the invariant tensors on Iw

$$\psi^+ = d(e^{56}), \quad \omega = e_7 \lrcorner \phi$$

are such that $d(*\omega) = 0$ and $d\omega = \psi^+$. It is no coincidence that this almost complex structure recalls the one investigated in [1] as a distinguished element in a ‘twistor space’ for Iw . The $SU(3)$ reduction (ω, ψ^+) is merely a modification of (2.1) obtained by rotations in the bundle $Iw \rightarrow T^4$ fibred by 2-tori, reminding of the Penrose fibration $\mathbb{C}\mathbb{P}^3 \rightarrow S^4$. A more systematic approach including this example was developed in [5].

3. A central chapter of the theory of exceptional geometry is related to Killing spinors [8]. This notion allowed Bär to prove that [7] if (X^6, g) is nearly Kähler, then the metric cone $(X^6 \times \mathbb{R}^+, t^2 g + dt^2)$ has holonomy G_2 .

4. Physical evidence has now shifted most of the concern towards metrics with orbifold singularities, see [2, 3, 6]. One with an isolated conical singularity (the most subtle of the three known in the simply-connected case) is the following. The space $X = S^3 \times S^3$ admits Einstein metrics, the easiest being the product of the two round metrics on the factors, that has symmetry $SU(2) \times SU(2) \times SU(2) \times SU(2)$. It has another Einstein – and here more relevant – metric invariant under $SU(2)^3 \times \Sigma_3$, where the latter is the symmetric group on 3 elements generating ‘triviality’. Describing X as the 3-symmetric space $SU(2)^3/SU(2)$ under the diagonal action, the metric is

$$g_X = -\text{tr}((a^{-1}da)^2 + (b^{-1}db)^2 + (c^{-1}dc)^2),$$

where $a = g_2g_3^{-1}$, $b = g_3g_1^{-1}$, $c = g_1g_2^{-1}$ and $(g_1, g_2, g_3) \in SU(2)^3$. The cone of g_X deforms to a smooth complete holonomy metric on some Y [12], itself homeomorphic to $\mathbb{R}^4 \times S^3$, because in the limit one of the spheres S^3 collapses. Thus Y has an asymptotically conical G_2 -metric.

5. The striking results achieved with the discovery of compact manifolds with holonomy G_2 by Joyce [25] first, and Kovalev [26] by different methods, answered the G_2 -analogue of the Calabi conjecture on special Hermitian holonomy. This is the origin of the expression *Joyce manifolds*. These constructions do not yield explicit metrics, though it must be said that at least for the purposes of string theorists, they need not necessarily be so.

4. SOLVABLE EXTENSIONS OF NILPOTENT LIE ALGEBRAS

Let N be now a 6-dimensional one-connected real nilpotent Lie group. Nilpotent means that there exists a basis (e^1, \dots, e^6) of left-invariant 1-forms on N such that

$$de^i \in \Lambda^2 \langle e^1, \dots, e^{i-1} \rangle, \quad i = 1, \dots, 6.$$

In terms of the lower central series $\mathfrak{n}^0 = \mathfrak{n}$, $\mathfrak{n}^i = [\mathfrak{n}^{i-1}, \mathfrak{n}]$, this is the same as requiring $\mathfrak{n}^s = 0$ for some $s \in \mathbb{N}$. If N has rational structure constants, then by [28] it admits a compact quotient $\Gamma \backslash N$ by a uniform discrete subgroup. Such a homogeneous space is called a nilmanifold.

Solvable extensions of nilpotent Lie groups are particular examples of homogeneous Einstein spaces of negative scalar curvature. All known non-compact, non-flat, homogeneous Einstein spaces have the form (S, g) , where S is a solvable Lie group and g is a left-invariant metric, which we will indicate by the name *solvmanifold*. Because left-invariant Einstein metrics on unimodular solvable Lie groups are flat [17], the solvable Lie groups we consider will be not unimodular, hence never admit a compact quotient [29]. The Einstein solvmanifolds available as of today are modelled on completely solvable Lie groups – the eigenvalues of ad_U are real, for any vector U – and their underlying metric Lie algebras $(\mathfrak{s}, \langle, \rangle)$ are *standard* and *of Iwasawa type*. Given a metric nilpotent Lie algebra $(\mathfrak{n}, [\cdot, \cdot]_{\mathfrak{n}}, \langle, \rangle')$ with inner product \langle, \rangle' , a metric solvable Lie algebra $(\mathfrak{s} = \mathfrak{n} \oplus \mathfrak{a}, [\cdot, \cdot], \langle, \rangle)$ is called a metric solvable extension of $(\mathfrak{n}, \langle, \rangle')$ if $[\cdot, \cdot]$ restricted to \mathfrak{n} coincides with $[\cdot, \cdot]_{\mathfrak{n}}$ and $\langle, \rangle|_{\mathfrak{n} \times \mathfrak{n}} = \langle, \rangle'$. One says that \mathfrak{s} is standard if $\mathfrak{a} = [\mathfrak{s}, \mathfrak{s}]^{\perp}$ is Abelian. The dimension of \mathfrak{a} is called the algebraic rank of \mathfrak{s} .

If the rank is one, say $\mathfrak{a} = \langle A \rangle$, the extension is of Iwasawa type if

- (i) $ad_A \neq 0$ is self-adjoint with respect to $\langle \cdot, \cdot \rangle$, and
- (ii) $(ad_A)|_{\mathfrak{n}}$ is positive-definite.

By [22, 4.18] the study of standard Einstein metric solvable Lie algebras reduces to rank-one metric solvable extensions

$$(\mathfrak{s} = \mathfrak{n} \oplus \mathbb{R}H, \langle \cdot, \cdot \rangle)$$

for some \mathfrak{n} and H with $\langle H, \mathfrak{n} \rangle = 0$, $\|H\| = 1$. The extended Lie bracket follows the rule

$$\begin{cases} [H, X] = D(X), \\ [X, Y] = [X, Y]_{\mathfrak{n}} \end{cases}$$

and $D \in \text{Der}(\mathfrak{n})$ is a derivation of the Lie algebra. If the metric $\langle \cdot, \cdot \rangle$ on the extension \mathfrak{s} is Einstein, then the derivation D is necessarily self-adjoint, and in fact unique [22]. For an Einstein solvmanifold, Heber calls eigenvalue type the sequence $(\lambda_1 < \dots < \lambda_r; m_1, \dots, m_r)$, where (λ_i) are the eigenvalues of D and (m_i) the corresponding multiplicities. He proved that in any dimension only finitely many eigenvalue types occur. In addition, six is a critical dimension, since by [27, 34] all nilpotent Lie groups up to dimension 6 admit an extension of rank one carrying an Einstein metric.

5. THE HOMOGENEOUS MODELS

From now Y will indicate a manifold equipped with the conformally parallel G_2 structure (2.3), so that the holonomy group of the metric $e^{2f}g$ is contained in G_2 . The function f is prescribed by $df = -me^7, m \in \mathbb{R}^-$.

In order to use the underlying almost Hermitian geometry, we suppose that Y arises from a rank-one solvable extension \mathfrak{s} of a metric nilpotent Lie algebra \mathfrak{n} , whose Lie group N is endowed with an invariant $SU(3)$ structure (J, ω, ψ^+) and a non-singular self-adjoint derivation D , as in the Einstein case. In this way the algebraic structure of \mathfrak{s} blends in with the Riemannian geometry of Y . We require in fact that $(DJ)^2 = (JD)^2$, a condition that translates into nice features of string models [21]. Concretely, the solvable structure is defined by the nilpotent \mathfrak{n} and by taking

$$D = ad_{e_7}.$$

A classification result establishes that N cannot be arbitrary, even p -step nilpotent with $p = 1$ or 2 . Under the above assumptions in fact, we proved that

5.1. Theorem. [15] *Let $\mathfrak{s} = \mathfrak{n} \oplus \mathbb{R}e_7$ be a rank-one solvable extension determined by ad_{e_7} . Then the G_2 structure $\varphi = \omega \wedge e^7 + \psi^+$ defined on \mathfrak{s} is conformally parallel if and only if \mathfrak{n} is isomorphic to one of:*

$$\begin{aligned} & (0, 0, e^{15}, e^{25}, 0, e^{12}), \quad (0, e^{54}, e^{64}, 0, 0, 0), \quad (0, e^{54}, e^{15} + e^{64}, 0, 0, 0, 0), \\ & (0, 0, e^{15} + e^{64}, 0, 0, 0), \quad (0, e^{61} + e^{54}, e^{15} + e^{64}, 0, 0, 0), \quad (0, 0, e^{15}, 0, 0, 0), \\ & (0, 0, 0, 0, 0, 0). \end{aligned}$$

Explicitly, the Lie algebras \mathfrak{s} are listed in the Table that follows. The terms corresponding to the nilpotent part have been highlighted to make it easier to recognize the underlying \mathfrak{n} of Theorem 5.1. About the notation: the ‘differential’

expression $(0, 0, e^{15}, e^{25}, 0, e^{12}, 0)$ is a quick way of saying $[e_5, e_1] = e_3$, $[e_5, e_2] = e_4$, $[e_2, e_1] = e_6$ for the basis of \mathfrak{g} . This is because a general Lie algebra \mathfrak{g} of dimension n is either prescribed by a Lie bracket $[\cdot, \cdot]$, or by a differential map $d : \mathfrak{g}^* \rightarrow \Lambda^2 \mathfrak{g}^*$ which extends to give a complex

$$0 \rightarrow \mathfrak{g}^* \rightarrow \Lambda^2 \mathfrak{g}^* \rightarrow \Lambda^3 \mathfrak{g}^* \rightarrow \dots \rightarrow \Lambda^n \mathfrak{g}^* \rightarrow 0.$$

Solvable Lie algebras
$(-me^{17}, -me^{27}, -me^{37}, -me^{47}, -me^{57}, -me^{67}, 0)$
$(-\frac{2}{3}me^{17}, -me^{27}, -\frac{4}{3}me^{37} + \frac{2}{3}me^{15}, -me^{47}, -\frac{2}{3}me^{57}, -me^{67}, 0)$
$(-\frac{3}{4}me^{17}, -me^{27}, -\frac{3}{2}me^{37} + \frac{1}{2}m(e^{15} - e^{46}), -\frac{3}{4}me^{47}, -\frac{3}{4}me^{57}, -\frac{3}{4}me^{67}, 0)$
$(-\frac{4}{5}me^{17}, -\frac{6}{5}me^{27} - \frac{2}{5}me^{45}, -\frac{7}{5}me^{37} + \frac{2}{5}m(e^{15} - e^{46}), -\frac{3}{5}me^{47}, -\frac{3}{5}me^{57}, -\frac{4}{5}me^{67}, 0)$
$(-me^{17}, -\frac{5}{4}me^{27} - \frac{1}{2}me^{45}, -\frac{5}{4}me^{37} - \frac{1}{2}me^{46}, -\frac{1}{2}me^{47}, -\frac{3}{4}me^{57}, -\frac{3}{4}me^{67}, 0)$
$(-\frac{2}{3}me^{17}, -\frac{4}{3}me^{27} - \frac{1}{3}m(e^{16} + e^{45}), -\frac{4}{3}me^{37} + \frac{1}{3}m(e^{15} - e^{46}), -\frac{2}{3}me^{47}, -\frac{2}{3}me^{57}, -\frac{2}{3}me^{67}, 0)$
$(-\frac{3}{5}me^{17}, -\frac{3}{5}me^{27}, -\frac{6}{5}me^{37} + \frac{2}{5}me^{15}, -\frac{6}{5}me^{47} + \frac{2}{5}me^{25}, -\frac{3}{5}me^{57}, -\frac{6}{5}me^{67} + \frac{2}{5}me^{12}, 0)$

The construction of Section 4 is particularly interesting since the results of [22, 34] ensure that Y will admit a homogeneous Einstein metric with negative scalar curvature, and moreover a unique one if one chooses the eigenvalues of ad_{e_7} .

5.1. Example of an Einstein metric. To end this section, we provide one of the Einstein metrics. Consider the 3-step solvable Lie algebra with structure equations

$$\begin{aligned} de^1 &= 2be^{17} + \sqrt{6}be^{26} \\ de^i &= be^{i7}, \quad i = 2, 4, 6 \\ de^3 &= 2be^{37} - \sqrt{6}be^{46} \\ de^5 &= 2be^{57} - \sqrt{6}be^{24} \\ de^7 &= 0, \end{aligned} \tag{5.1}$$

where b is real and not zero. This Lie algebra is actually the last one in the Table in disguise, endowed with the 3-form (2.3). The G_2 structure satisfies the conditions

$$\begin{aligned} d\varphi &= 5b(e^{1257} + e^{1367} - e^{3457}) - 3b(\sqrt{6} - 1)e^{2467}, \\ d(*\varphi) &= \sqrt{6}b(\sqrt{6} + 1)e^{23567} + \sqrt{6}b(\sqrt{6} - 1)(e^{12347} - e^{14567}), \end{aligned}$$

so it belongs to the class $\mathcal{T}_1 \oplus \mathcal{T}_3$ and the associated metric $\sum_{i=1}^7 (e^i)^2$ is Einstein with Ricci tensor $Ric(g) = -15b^2g$. We shall return to this example at the very end of this survey.

6. RICCI-FLAT METRICS

Another aspect of the picture is that the almost Hermitian manifold N is half-flat [16]. If one considers a 6-manifold N equipped with a reduction (ω, ψ^+) that

depends on a real parameter $t \in I$, let's say a 'time-dependent' $SU(3)$ -structure, then $N^6 \times I$ is a warped G_2 manifold with fundamental form

$$\varphi = \omega \wedge dt + \psi^+.$$

If $(N^6 \times I, \varphi)$ is parallel, the forms (ω^2, ψ^+) evolve according to differential equations

$$d\omega = \frac{\partial \psi^+}{\partial t}, \quad d(J\psi^+) = -\frac{\partial}{\partial t}(\frac{1}{2}\omega^2), \quad (6.1)$$

coming from the Hamiltonian flow of a functional. The opposite is also true. If N is compact and (6.1) are satisfied by closed forms ψ^+ and ω^2 of suitable algebraic type, there exists a metric with holonomy contained in G_2 on the product of N^6 with some interval I [23].

The system (6.1) is tough to solve in general. To apply Hitchin's theorem, instead of considering a nilpotent Lie group N we work with the associated nilmanifold and use the left-invariance of the forms. The Ricci-flat metrics thus found were described in [15] and coincide with homogeneous metrics possessing a conformal Killing field. This is attained by comparing the expressions, and bearing in mind that the simply-connected solvable Lie group S (corresponding to \mathfrak{s}) is diffeomorphic to \mathbb{R}^7 , hence admits global coordinates of type (x_1, \dots, x_6, t) . At the same time the metric can be seen as living on the product $\mathbb{R} \times \Gamma \backslash N$, where the nilmanifold $\Gamma \backslash N$ is 2-step nilpotent, or T^6 . This explains the bundle structure appearing, since [30] torus fibrations over tori are, essentially, nilmanifolds of step-length two. The isometry between the two metrics is given by an appropriate choice of frame (e^i) , whence

$$g = e^{-2mt} \sum_{i=1}^7 (e^i)^2 \quad (6.2)$$

has a subgroup of G_2 as holonomy.

7. TWO EXAMPLES WITH FULL HOLONOMY

We carry out some calculations showing how the G_2 -holonomy metrics are related to the Ricci-flat ones.

7.1. First example. It is clear that the Lie algebra

$$\left(-\frac{4}{5}me^{17}, -\frac{6}{5}me^{27} - \frac{2}{5}me^{45}, -\frac{7}{5}me^{37} + \frac{2}{5}m(e^{15} - e^{46}), -\frac{3}{5}me^{47}, -\frac{3}{5}me^{57}, -\frac{4}{5}me^{67}, 0\right)$$

extends the nilpotent Lie algebra with non-zero brackets

$$e_2 = [e_5, e_4], \quad e_3 = [e_6, e_4] = [e_1, e_5].$$

Consider the Riemannian metric

$$g = e^{-2mt} dt^2 + e^{-\frac{2}{5}mt} (dx_1^2 + dx_6^2) + e^{-\frac{4}{5}mt} (dx_4^2 + dx_5^2) + \frac{9}{25} m^2 e^{\frac{4}{5}mt} (dx_3 - \frac{2}{3}x_1 dx_5 + \frac{2}{3}x_4 dx_6)^2 + \frac{9}{25} m^2 e^{\frac{2}{5}mt} (dx_2 + \frac{2}{3}x_4 dx_5)^2, \quad (7.1)$$

whose holonomy group is precisely G_2 .

7.1. Proposition. *This g is locally isometric to*

$$ds^2 = V^3 dw^2 + V(du_1^2 + du_4^2) + V^2(du_2^2 + du_3^2) + \\ V^{-2}(dy_2 + k(u_2 du_4 - u_1 du_3))^2 + V^2(dy_1 + ku_2 du_3)^2,$$

on the product of $\mathbb{R} \times \mathcal{B}$ where \mathcal{B} is the total space of a torus bundle

$$\mathcal{B} \longrightarrow T^4.$$

Proof. By demanding that

$$e^1 = e^{\frac{4}{5}mt} dx_1, \quad e^2 = -\frac{3}{5}m e^{\frac{6}{5}mt} (dx_2 + \frac{2}{3}x_4 dx_5), \\ e^3 = -\frac{3}{5}m e^{\frac{7}{5}mt} (dx_3 - \frac{2}{3}x_1 dx_5 + \frac{2}{3}x_4 dx_6), \quad e^4 = e^{\frac{3}{5}mt} dx_4, \quad (7.2) \\ e^5 = e^{\frac{3}{5}mt} dx_5, \quad e^6 = e^{\frac{4}{5}mt} dx_6, \quad e^7 = dt$$

be an orthonormal frame for $\mathfrak{n} \oplus \mathbb{R}$, we have that g as in (6.2) recovers (7.1). The local equivalence is established once we indicate by (u_i) the coordinates on T^4 , by y_1, y_2 those on the fibres T^2 , and have w describe the seventh direction, with $V = mw$. \square

7.2. Theorem. *The family of Ricci-flat metrics arising from the $SU(3)$ structures solutions of (6.1) essentially coincides with (7.1).*

Proof. We deform the starting $SU(3)$ reduction

$$\frac{1}{2}\omega_0 \wedge \omega_0 = -e^{2356} - e^{1423} + e^{1456}, \quad \psi_0^+ = e^{125} - e^{345} + e^{136} + e^{246} \quad (7.3)$$

determined by (2.1) by forms on \mathfrak{n} representing zero cohomology, so that

$$([\omega_0^2], [\psi_0^+]) = ([\omega(t)^2], [\psi^+(t)]) \quad \text{in } H^4(\Gamma \backslash N, \mathbb{R}) \times H^3(\Gamma \backslash N, \mathbb{R}) \quad \text{for all } t\text{'s.}$$

The four-form $\omega_0 \wedge \omega_0$ flows under (6.1) according to

$$\frac{1}{2}\omega^2(t) = -P(t)(e^{1423} + e^{2356}) + (P(t) + D(t))e^{1456}$$

for smooth maps P, D with $P(0) = 1, D(0) = 0$. In the same way the three-form turns out to be

$$\psi^+(t) = (M(t) + 1)(e^{125} - e^{345} + e^{246}) + e^{136}, \quad M(0) = 1.$$

This almost gives the Kähler form as

$$\omega(t) = \sqrt{P + D}(e^{14} + e^{56}) + \frac{5M'}{2m}e^{23},$$

with the dash denoting derivatives with respect to t . Notice how the expression respects the bundle structure of \mathcal{B} . At this point it is anybody's guess to solve (6.1), because one does not know $\psi^-(t)$, which normally makes the system extremely hard to tackle. To this end we define an orthonormal basis $(\lambda^{a_i} e^i)$, in which the choice of exponents a_i ought to reflect the form of $\psi^+(t)$ above. Picking

$$a_i = (-1, 1, 2, -2, -2, -1),$$

for example, one has

$$\begin{aligned}\psi^+ &= \lambda^{-2}(e^{125} - e^{345} + e^{246}) + e^{136}, \\ \psi^- &= \lambda^{-2}e^{126} - \lambda^{-4}e^{245} - \lambda^{-3}e^{135} - \lambda^{-1}e^{346}\end{aligned}$$

giving

$$P = 1, \quad D' = \frac{6}{5}m\sqrt{M+1}$$

with

$$\lambda(t)^{-2} = M + 1.$$

Since $\psi^+ \wedge \psi^-$ and ω^3 are both volume forms on N , the Cauchy system is solved by

$$M(t) = (1 - mt)^{2/5} - 1.$$

It is then a simple matter to write the induced metric $g(t)$

$$\begin{aligned}(1 - mt)^{2/5} ((e^1)^2 + (e^6)^2) + (1 - mt)^{4/5} ((e^4)^2 + (e^5)^2) \\ + (1 - mt)^{-2/5} (e^2)^2 + (1 - mt)^{-4/5} (e^3)^2 + dt^2\end{aligned}$$

and see that this is (7.1), provided one rescales time $t \mapsto e^{-mt}t$ and changes names to the variables with the recipe (7.2). \square

For a better understanding of the process one could also reconsider it within its symplectic framework [23]. The natural variables $p(t), q(t)$ of the candidate Hamiltonian function $H(t) = H(p, q)$ have to satisfy the standard relations

$$\begin{cases} p' = -\frac{\partial H}{\partial q} \\ q' = \frac{\partial H}{\partial p} \end{cases}.$$

They translate here into

$$p' = -\frac{2}{5}m\sqrt{q+1} \quad q' = \frac{6}{5}m\sqrt{p+1},$$

leading to

$$6((p+1)^{3/2} - 1) = 1 - (q+1)^{3/2}. \quad (7.4)$$

Therefore the functions are

$$p(t) = (1 - mt)^{2/5} - 1 \quad \text{and} \quad q(t) = \frac{6}{5}m(1 - mt)^{1/2}$$

and H , constant on the level curves of (7.4), is

$$H(t) = \frac{2}{5}m(1 - mt)^{9/5} + \frac{6}{5}m(1 - mt)^{3/5}.$$

7.2. Second example. The Lie algebra

$$\left(-\frac{3}{5}me^{17}, -\frac{3}{5}me^{27}, \frac{2}{5}me^{15} - \frac{6}{5}me^{37}, \frac{2}{5}me^{25} - \frac{6}{5}me^{47}, -\frac{3}{5}me^{57}, \frac{2}{5}me^{12} - \frac{6}{5}me^{67}, 0\right),$$

arises from $\mathfrak{n} = (0, 0, \frac{2}{5}me^{15}, \frac{2}{5}me^{25}, 0, \frac{2}{5}me^{12})$ and is isomorphic to (5.1). Consider the following metric

$$g = e^{-\frac{4}{5}mt}(dx_1^2 + dx_2^2 + dx_5^2) + \frac{9}{25}m^2e^{\frac{2}{5}mt}(dx_3 + \frac{2}{3}x_5dx_1)^2 \\ + \frac{9}{25}m^2e^{\frac{2}{5}mt}(dx_4 - \frac{2}{3}x_2dx_5)^2 + \frac{9}{25}m^2e^{\frac{2}{5}mt}(dx_6 + \frac{2}{3}x_2dx_1)^2 + e^{-2mt}dt^2 \quad (7.5)$$

with the identifications

$$\begin{cases} e^i = e^{\frac{3}{5}mt}dx_i, & i = 1, 2, 5, \\ e^3 = -\frac{1}{5}me^{\frac{6}{5}mt}(3dx_3 + 2x_5dx_1), \\ e^4 = -\frac{1}{5}me^{\frac{6}{5}mt}(3dx_4 - 2x_2dx_5), \\ e^6 = -\frac{1}{5}me^{\frac{6}{5}mt}(3dx_6 + 2x_2dx_1), \\ e^7 = dt. \end{cases}$$

These expressions make g a G_2 -holonomy metric on $S = \mathbb{R} \times \mathcal{D}$, \mathcal{D} being the the T^3 -bundle over the torus T^3 associated to $\text{span}\{e_3, e_4, e_6\}$.

7.3. Theorem. *The nilpotent Lie algebra $(0, 0, \frac{2}{5}me^{15}, \frac{2}{5}me^{25}, 0, \frac{2}{5}me^{12})$ equipped with $SU(3)$ forms $\omega_0 = e^{56} - e^{23} + e^{14}$, $\psi_0^+ = -e^{345} + e^{136} + e^{246} + e^{125}$ generates the Ricci-flat metric*

$$g = (1 - mt)^{4/5}g_{\text{fibre}} + (1 - mt)^{-2/5}g_{\text{base}} + dt^2$$

on S , in terms of the flat metrics $g_{\text{fibre}} = (e^1)^2 + (e^2)^2 + (e^5)^2$ and $g_{\text{base}} = (e^3)^2 + (e^4)^2 + (e^6)^2$ on \mathcal{D} .

Proof. The square of ω_0 is an exact form, as $d(e^{364}) = \frac{1}{2}\omega_0^2$, whereby

$$\omega^2(t) = P(t)\omega_0^2$$

for some smooth function P on an interval $I \subseteq \mathbb{R}$ such that $P(0) = 1$. As for ψ^+ , the boundary conditions ensure that just the term e^{125} varies. Equations (6.1) together with the primitivity of ψ^+ (holding at all time) yield

$$\begin{cases} \psi^+(t) = -e^{345} + e^{136} + e^{246} + (E + 1)e^{125} \\ \omega(t) = \pm\sqrt{P}\omega_0 \end{cases}$$

with $E(0) = 1$. The solution $P(t) = (1 - \frac{5}{2}t)^{2/5}$ implies that only the volume of the fibres intervenes in the evolution of the three-form

$$\psi^+ = (1 - mt)^{6/5}e^{125} + e^{136} + e^{246} - e^{345},$$

whilst the Kähler structure deforms as

$$\omega = (1 - mt)^{1/5}(e^{14} - e^{23} + e^{56}).$$

By writing the compatible metric one obtains the desired expression. The fibres grow at double the speed at which the base shrinks, precisely as in (7.5). \square

The reader might want to compare at this point the horizontal/vertical split of this metric to the similar one of (3.2).

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