REVISTA DE LA
UNIÓN MATEMÁTICA ARGENTINA
Volumen 47, Número 1, 2006, Páginas 85–98

ON COMPLETE SPACELIKE SUBMANIFOLDS IN THE DE SITTER SPACE WITH PARALLEL MEAN CURVATURE VECTOR

ROSA MARIA S. BARREIRO CHAVES AND LUIZ AMANCIO M. SOUSA JR.

Abstract. The text surveys some results concerning submanifolds with parallel mean curvature vector immersed in the De Sitter space. We also propose a semi-Riemannian version of an important inequality obtained by Simons in the Riemannian case and apply it in order to obtain some results characterizing umbilical submanifolds and a product of submanifolds in the $(n+p)$-dimensional De Sitter space $\mathbb{S}_p^{n+p}$.

1. Introduction

Let $\mathbb{R}_p^{n+p+1}$ be an $(n+p+1)$-dimensional real vector space endowed with an inner product of index p given by

$$< x, y >= -\sum_{i=1}^{p} x_i y_i + \sum_{j=p+1}^{n+p+1} x_j y_j,$$

where $x = (x_1, x_2, \cdots, x_{n+p+1})$ is the natural coordinate of $\mathbb{R}_p^{n+p+1}$.

We also define the semi-Riemannian manifold $\mathbb{S}_p^{n+p}$, by

$$\mathbb{S}_p^{n+p} \quad = \{(x_1, x_2, \cdots x_{n+p+1}) \in \mathbb{R}_p^{n+p+1} \,/ -\sum_{i=1}^{p} {x_i}^2 + \sum_{j=p+1}^{n+p+1} {x_j}^2 = 1\} \; .$$

$\mathbb{S}_p^{n+p}$ is called $(n+p)$-*dimensional De Sitter space of index* p.

Let M^n be an n-dimensional semi-Riemannian manifold immersed in $\mathbb{S}_p^{n+p}$. M^n is said to be *spacelike* if the induced metric on M^n from the metric of $\mathbb{S}_p^{n+p}$ is positive definite.

From now on, we will consider spacelike submanifolds M^n of $\mathbb{S}_p^{n+p}$ with parallel mean curvature vector h. Let $H = |h|$ be the mean curvature of M^n. If h is parallel it is easy to verify that H is constant and, when $p = 1$, these two conditions are equivalent. We say that M^n is a maximal submanifold if h vanishes identically.

It was proved by E. Calabi [6] (for $n \leq 4$) and by S.Y. Cheng and S.T. Yau [8] (for all n) that a complete maximal spacelike hypersurface in $\mathbb{R}_1^{n+1}$ is totally

2000 *Mathematics Subject Classification.* Primary 53C42, 53A10.
*Key words and phrases.*De Sitter space, Simons type formula, complete spacelike submanifolds, parallel mean curvature vector.

geodesic. In [17], S. Nishikawa obtained similar results for others Lorentzian manifolds. In particular, he proved that a complete maximal spacelike hypersurface in $\mathbb{S}_1^{n+1}$ is totally geodesic. We recall that a submanifold M^n is said totally geodesic if its second fundamental form B vanishes identically.

A. Goddard [11] conjectured that the complete spacelike hypersurfaces of $\mathbb{S}_1^{n+1}$ with H constant must be totally umbilical. The totally umbilical hypersurfaces of $\mathbb{S}_1^{n+1}$ are obtained by intersecting $\mathbb{S}_1^{n+1}$ with linear hyperplanes through the origin of $\mathbb{R}_1^{n+2}$, where $\mathbb{S}_1^{n+1}$ can be viewed as hypersphere of $\mathbb{R}_1^{n+2}$.

J. Ramanathan [19] proved Goddard's conjecture for $\mathbb{S}_1^3$ and $0 \leq H \leq 1$. Moreover, if $H > 1$ he showed that the conjecture is false as can be seen from an example due to Dajczer-Nomizu [10]. In his proof, Ramanthan used the complex structure of $\mathbb{S}_1^3$. K. Akutagawa [2] proved that Goddard's conjecture is true when $n = 2$ and $H^2 \leq 1$ or when $n \geq 3$ and $H^2 < \frac{4(n-1)}{n^2}$. He also constructed complete spacelike rotation surfaces in $\mathbb{S}_1^3$ with constant H satisfying $H > 1$ and which are not totally umbilical.

In [15], S. Montiel proved that Goddard's conjecture is true provided that M^n is compact. Furthermore, he exhibited examples of complete spacelike hypersurfaces with constant H satisfying $H^2 \geq \frac{4(n-1)}{n^2}$ and being not totally umbilical - the so called hyperbolic cylinders (cf. [2] and [13]), which are isometric to the Riemannian product $\mathbb{H}^1(\sinh r) \times \mathbb{S}^{n-1}(\cosh r)$ of a hyperbolic line and an $(n-1)$-dimensional sphere of constant seccional curvatures $1 - \coth^2 r$ and $1 - \tanh^2 r$, respectively. Later, Montiel [16] studied complete spacelike hypersurfaces with constant mean curvature $H^2 = \frac{4(n-1)}{n^2}$ and proved the following result.

Theorem 1.1. *Let M^n be a complete spacelike hypersurfaces in $\mathbb{S}_1^{n+1}$ with constant mean curvature $H^2 = \frac{4(n-1)}{n^2}$. If M^n is not connected at infinity, that is, if M^n has at least two ends, then M^n is, up to isometry, a hyperbolic cylinder.*

Concerning to submanifolds M^n of $\mathbb{S}_p^{n+p}$ with parallel mean curvature vector we may cite the following remarkable results. In [12], T. Ishihara proved the following theorem that generalizes for higher codimension the result of Cheng-Yau [8]

Theorem 1.2. *Let M^n be an n-dimensional complete Riemannian manifold isometrically immersed in $\mathbb{R}_p^{n+p}$ or $\mathbb{S}_p^{n+p}$. If M^n is maximal, then the immersion is totally geodesic and M^n is a Riemannian space of constant curvature.*

In [7], Q.M. Cheng showed that Akutagawa's result [2] is valid for higher codimensional complete spacelike submanifolds in $\mathbb{S}_p^{n+p}$ with parallel mean curvature vector. More precisely, he proved the following result.

Theorem 1.3. *Let M^n be an n-dimensional complete spacelike submanifold in $\mathbb{S}_p^{n+p}$ with parallel mean curvature vector. If $H^2 \leq 1$, when n=2 or $n^2H^2 < 4(n-1)$, when $n \geq 3$, then M^n is totally umbilical.*

In [14], H. Li obtained the following extension of Theorem 1.1.

Theorem 1.4. *Let M^n be an n-dimensional complete spacelike submanifold in $\mathbb{S}_p^{n+p}$ with parallel mean curvature vector. If $H^2 = \frac{4(n-1)}{n^2}$ and M^n is not connected* *at infinity, that is, if M^n has at least two ends, then M^n is, up to isometry, a hyperbolic cylinder in $\mathbb{S}_1^{n+1}$.*

R. Aiyama [1] studied compact spacelike submanifold in $\mathbb{S}_p^{n+p}$ with parallel mean curvature vector and proved the following results:

Theorem 1.5. *Let M^n be an n-dimensional compact spacelike submanifold in $\mathbb{S}_p^{n+p}$ with parallel mean curvature vector. If the normal connection of M^n is flat, then M^n is totally umbilical.*

Theorem 1.6. *Let M^n be an n-dimensional compact spacelike submanifold in $\mathbb{S}_p^{n+p}$ with parallel mean curvature vector. If the sectional curvature of M^n is non-negative, then M^n is totally umbilical.*

We point out that L. Alias and A. Romero [3] also obtained results related to complete spacelike submanifolds in $\mathbb{S}_p^{n+p}$ with parallel mean curvature vector.

Let $\mathbb{S}^n(r)$ be an n-dimensional sphere in $\mathbb{R}^{n+1}$ with radius r and let M^n be an n-dimensional submanifold minimally immersed in $\mathbb{S}^{n+p}(1)$. Denote by B the second fundamental form of this immersion and by S the square of the length of B. In his pioneering work, J. Simons [20] proved the following inequality for ΔS

$$\frac{1}{2}\Delta S \geq S\left(n - \left(2 - \frac{1}{p}\right) S\right). \tag{1.1}$$

As an application of formula (1.1), Simons [20] obtained the following result.

Theorem 1.7. *Let M^n be a closed minimal submanifold of $\mathbb{S}^{n+p}(1)$. Then either M^n is totally geodesic, or $S = \frac{n}{2-\frac{1}{p}}$, or $\sup S > \frac{n}{2-\frac{1}{p}}$.*

Two years later, S.S. Chern, M. do Carmo and S. Kobayashi [9], determined all the minimal submanifolds of $\mathbb{S}^{n+p}(1)$ satisfying $S = \frac{n}{2-\frac{1}{p}}$. More precisely, they proved:

Theorem 1.8. *Let M^n be a closed minimal submanifold of $\mathbb{S}^{n+p}(1)$. Assume that $S \leq \frac{n}{2-\frac{1}{p}}$. Then:*

(i) Either $S = 0$ (and M^n is totally geodesic) or $S = \frac{n}{2-\frac{1}{p}}$.

(ii) $S = \frac{n}{2-\frac{1}{p}}$ if and only if:

a) $p = 1$ and M^n is locally a Clifford torus $\mathbb{S}^k\left(\sqrt{\frac{k}{n}}\right) \times \mathbb{S}^{n-k}\left(\sqrt{\frac{n-k}{n}}\right)$.

b) $p = n = 2$ and M^2 is locally a Veronese surface in $\mathbb{S}^4(1)$.

In the case of a submanifold M^n of $\mathbb{S}^{n+p}(1)$ with non-zero parallel mean curvature vector h, it is convenient to modify slightly the second fundamental form B and to introduce the tracelless tensor $\Phi = B - Hg$, where $H = \mid h \mid$ is the mean curvature and g stands for the induced metric on M^n. W. Santos [21] established the following inequality for the Laplacian of $\mid \Phi \mid^2$

$$\frac{1}{2}\Delta \mid \Phi \mid^2 \geq \mid \Phi \mid^2 \left(n(1 + H^2) - \frac{n(n-2)}{\sqrt{n(n-1)}} \mid g(\Phi, h) \mid - \left(\frac{2p-3}{p-1}\right) \mid \Phi \mid^2\right).$$

Let M^n be a complete spacelike maximal submanifold of $\mathbb{S}_p^{n+p}$. In [12], T. Ishihara derived the following inequality for ΔS

$$\frac{1}{2}\Delta S \geq S\left(n+\frac{S}{p}\right). \tag{1.2}$$

As an important application of (1.2), Ishihara proved Theorem 1.2.

If M^n is a spacelike hypersurface of $\mathbb{S}_1^{n+1}$ with constant mean curvature H, as in the Riemannian case, it is convenient to consider the tensor Φ. U.H. Ki, H.J. Kim and H. Nakagawa [13], established the following inequality for $\Delta \mid \Phi \mid^2$

$$\frac{1}{2}\Delta \mid \Phi \mid^2 \geq \mid \Phi \mid^2 \left(\mid \Phi \mid^2 - \frac{n(n-2)}{\sqrt{n(n-1)}} H \mid \Phi \mid + n(1-H^2)\right). \tag{1.3}$$

By applying (1.3) they obtained a constant S_+ that depends on n and H and such that $S \leq S_+$. They also characterized the hyperbolic cylinders as the only complete spacelike hypersurfaces of $\mathbb{S}_1^{n+1}$ with non-zero constant H and $S = S_+$. Moreover, they proved that a complete spacelike hypersurface of $\mathbb{S}_1^{n+1}$ with non-zero constant H and non-negative sectional curvature is totally umbilical, provided that $S < S_+$.

A. Brasil, G. Colares and O. Palmas [5] obtained the following gap theorem.

Theorem 1.9. *Let M^n, $n \geq 3$, be a complete spacelike hypersurface in $\mathbb{S}_1^{n+1}$ with constant mean curvature $H > 0$. Then sup $\mid \Phi \mid^2 < \infty$ and*
a) either sup $\mid \Phi \mid = 0$ and M^n is totally umbilical or
b) $B_H^- \leq \sqrt{sup \mid \Phi \mid^2} \leq B_H^+$, where $B_H^- \leq B_H^+$ are the roots of the polynomial

$$P_H(x) = x^2 - \frac{n(n-2)}{\sqrt{n(n-1)}} Hx + n(1-H^2).$$

Recently, A. Brasil, R.M.B. Chaves and G. Colares [4] extended the above result for complete spacelike submanifolds in $\mathbb{S}_p^{n+p}$ with parallel mean curvature vector.

Let M^n be a spacelike submanifold of $Q_p^{n+p}(c)$ with non-zero parallel mean curvature vector h and let $H = \mid h \mid$. Define the second fundamental form with respect to the normal direction $\xi = \frac{h}{H}$ by h^ξ. If $\mid h^\xi \mid^2$ denotes the squared norm of h^ξ, set $\mid \mu \mid^2 = \mid h^\xi \mid^2 - nH^2$. In [7], Q. M. Cheng proved that

$$\frac{1}{2}\Delta \mid \mu \mid^2 \geq \mid \mu \mid^2 \left(\mid \mu \mid^2 - \frac{n(n-2)}{\sqrt{n(n-1)}} H \mid \mu \mid + n(1-H^2)\right). \tag{1.4}$$

Now we are going to state our main results. Theorem 1.10 is a Simons' type inequality for submanifolds in De Sitter space $\mathbb{S}_p^{n+p}$.

Theorem 1.10. *Let M^n be a spacelike submanifold immersed in $\mathbb{S}_p^{n+p}$ with parallel mean curvature. Then the following inequality holds*

$$\tfrac{1}{2}\Delta \mid \Phi \mid^2 \quad \geq \mid \Phi \mid^2 \left(\frac{\mid \Phi \mid^2}{p} - \frac{n(n-2)}{\sqrt{n(n-1)}} H \mid \Phi \mid + n(1-H^2)\right). \tag{1.5}$$

Next Theorem is a Lorentzian version of results obtained by K. Yano and S. Ishihara [22] and also by S.T. Yau [23] for Riemannian submanifolds.

Theorem 1.11. *Let M^n be a complete spacelike submanifold in $\mathbb{S}_p^{n+p}$ with parallel mean curvature vector and non-negative sectional curvature. If M^n has constant scalar curvature R, then M^n is totally umbilical or a product $M_1 \times M_2 \times \cdots \times M_k$, where each M_i is a totally umbilical submanifold of $\mathbb{S}_p^{n+p}$ and the M_i's are mutually perpendicular along their intersections.*

As we saw in the Theorem 1.6, compact spacelike submanifolds in $\mathbb{S}_p^{n+p}$ with parallel mean curvature vector and non-negative sectional curvature are totally umbilic.

The following result is an application of formula (1.5).

Theorem 1.12. *Let M^n be a complete spacelike submanifold in $\mathbb{S}_p^{n+p}$ with parallel mean curvature vector. If $\sup K$ denotes the function that assigns to each point of M^n the supremum of the sectional curvatures at that point, there exists a constant $\beta(n,p,H)$ such that if $\sup K \leq \beta(n,p,H)$, then either:*
(i) $n = 2$ and M^2 is totally umbilical or
(ii) $n \geq 3$ and M^n is totally geodesic.

2. Preliminaries

In this section we will introduce some basic facts and notations that will appear on the paper. Let M^n be an n-dimensional Riemannian manifold immersed in $\mathbb{S}_p^{n+p}$. As the indefinite Riemannian metric of $\mathbb{S}_p^{n+p}$ induces the Riemannian metric of M^n, the immersion is called spacelike. We choose a local field of semi-Riemannian orthonormal frames $e_1, ..., e_{n+p}$ in $\mathbb{S}_p^{n+p}$ such that, at each point of M^n, $e_1, ..., e_n$ span the tangent space of M^n. We make the following standard convention of indices

$$1 \leq A, B, C, \cdots \leq n+p, \; 1 \leq i, j, k, \cdots \leq n, \; n+1 \leq \alpha, \beta, \gamma, \cdots \leq n+p.$$

Take the correspondent dual coframe $\{\omega_1, ..., \omega_{n+p}\}$ such that the semi-Riemannian metric of $\mathbb{S}_p^{n+p}$ is given by $d\bar{s}^2 = \sum_i {\omega_i}^2 - \sum_\alpha {\omega_\alpha}^2 = \sum_A \varepsilon_A {\omega_A}^2, \varepsilon_i = 1, \varepsilon_\alpha = -1,$ $1 \leq i \leq n, n+1 \leq \alpha \leq n+p$. Then the structure equations of $\mathbb{S}_p^{n+p}$ are given by

$$d\omega_A = \sum_B \varepsilon_B \omega_{AB} \wedge \omega_B, \quad \omega_{AB} + \omega_{BA} = 0. \tag{2.1}$$

$$d\omega_{AB} = \sum_C \varepsilon_C \omega_{AC} \wedge \omega_{CB} - \frac{1}{2} \sum_{C,D} \varepsilon_C \varepsilon_D K_{ABCD} \omega_C \wedge \omega_D. \tag{2.2}$$

$$K_{ABCD} = \varepsilon_A \varepsilon_B (\delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC}). \tag{2.3}$$

Next, we restrict those forms to M^n. First of all we get

$$\omega_\alpha = 0, \quad n+1 \leq \alpha \leq n+p. \tag{2.4}$$

So the Riemannian metric of M^n is written as $ds^2 = \sum_i {\omega_i}^2$.

Since $0 = d\omega_\alpha = \sum_i \omega_{\alpha i} \wedge \omega_i$, from *Cartan's lemma*, we can write

$$\omega_{\alpha i} = \sum_j h^\alpha_{ij}\omega_j, \quad h^\alpha_{ij} = h^\alpha_{ji}. \tag{2.5}$$

Set $B = \sum_{\alpha,i,j} h^\alpha_{ij}\omega_i\omega_j e_\alpha$, $h = \frac{1}{n}\sum_\alpha \left(\sum_i h^\alpha_{ii}\right) e_\alpha$ and $H = |h| = \frac{1}{n}\sqrt{\sum_\alpha (\sum_i h^\alpha_{ii})^2}$ the *second fundamental form, the mean curvature vector* and *the mean curvature* of M^n, respectively.

Using the structure equations we obtain the *Gauss equation*

$$R_{ijkl} = (\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) - \sum_\alpha \left(h^\alpha_{ik}h^\alpha_{jl} - h^\alpha_{il}h^\alpha_{jk}\right). \tag{2.6}$$

The *scalar curvature* R is given by

$$R = n(n-1) - n^2H^2 + S, \tag{2.7}$$

where $S = \sum_{\alpha,i,j}(h^\alpha_{ij})^2$ is the squared norm of the second fundamental form of M^n.

We also have the structure equations of the normal bundle of M^n

$$d\omega_\alpha = \sum_\beta \omega_{\alpha\beta} \wedge \omega_\beta, \quad \omega_{\alpha\beta} + \omega_{\beta\alpha} = 0. \tag{2.8}$$

$$d\omega_{\alpha\beta} = \sum_\gamma \omega_{\alpha\gamma} \wedge \omega_{\gamma\beta} - \frac{1}{2}\sum_{i,j} R_{\alpha\beta ij}\omega_i \wedge \omega_j, \tag{2.9}$$

where

$$R_{\alpha\beta ij} = \sum_l \left(h^\alpha_{il}h^\beta_{lj} - h^\alpha_{jl}h^\beta_{li}\right). \tag{2.10}$$

The covariant derivatives h^α_{ijk} of h^α_{ij} satisfy

$$\sum_k h^\alpha_{ijk}\omega_k = dh^\alpha_{ij} + \sum_k h^\alpha_{ik}\omega_{kj} + \sum_k h^\alpha_{jk}\omega_{ki} - \sum_\beta h^\beta_{ij}\omega_{\beta\alpha}. \tag{2.11}$$

Then, by exterior differentiation of (2.5), we obtain the *Codazzi equation*

$$h^\alpha_{ijk} = h^\alpha_{jik} = h^\alpha_{ikj}. \tag{2.12}$$

Similarly, we have the second covariant derivatives h^α_{ijkl} of h^α_{ij} so that

$$\begin{aligned}\sum_l h^\alpha_{ijkl}\omega_l = dh^\alpha_{ijk} + \sum_l h^\alpha_{ljk}\omega_{li} + \sum_l h^\alpha_{ilk}\omega_{lj} + \\ \sum_l h^\alpha_{ijl}\omega_{lk} - \sum_\beta h^\beta_{ijk}\omega_{\beta\alpha}.\end{aligned} \tag{2.13}$$

By exterior differentiation of (2.11), we can get the following *Ricci formula*

$$h^\alpha_{ijkl} - h^\alpha_{ijlk} = \sum_m h^\alpha_{im}R_{mjkl} + \sum_m h^\alpha_{jm}R_{mikl} + \sum_\beta h^\beta_{ij}R_{\alpha\beta kl}. \tag{2.14}$$

The Laplacian $\triangle h^\alpha_{ij}$ of h^α_{ij} is defined by $\triangle h^\alpha_{ij} = \sum_k h^\alpha_{ijkk}$. From (2.12) and (2.14), we have

$$\triangle h^\alpha_{ij} = \sum_k h^\alpha_{kkij} + \sum_{m,k} h^\alpha_{km} R_{mijk} + \sum_{m,k} h^\alpha_{mi} R_{mkjk} + \sum_{k,\beta} h^\beta_{ik} R_{\alpha\beta jk}. \tag{2.15}$$

If $H \neq 0$, we choose $e_{n+1} = \dfrac{h}{H}$. Thus

$$H^{n+1} = \frac{1}{n}\mathrm{tr}h^{n+1} = H \text{ and } H^\alpha = \frac{1}{n}\mathrm{tr}h^\alpha = 0, \alpha \geq n+2, \tag{2.16}$$

where h^α denotes the matrix $[h^\alpha_{ij}]$.

From (2.6), (2.10), (2.15) and (2.16) it is straightforward to verify that

$$\begin{aligned} \frac{1}{2}\triangle S = & \sum_{\alpha,i,j,k} (h^\alpha_{ijk})^2 + n \sum_{\alpha,i,j} h^\alpha_{ij} H^\alpha_{ij} + \\ & (nS - n^2H^2) - nH \sum_\alpha tr(h^{n+1}(h^\alpha)^2) + \\ & \sum_{\alpha,\beta} [tr(h^\alpha h^\beta)]^2 + \sum_{\alpha,\beta} N(h^\alpha h^\beta - h^\beta h^\alpha), \end{aligned} \tag{2.17}$$

where $N(A) = tr(AA^t)$, for all matrix $A = [a_{ij}]$.

Recall that M^n is a submanifold with parallel mean curvature vector h if $\nabla^\perp h \equiv 0$, where $\nabla^\perp$ is the normal connection of M^n in $\mathbb{S}^{n+p}_p$. Note that this condition implies that $H = |h|$ is constant and

$$\sum_k h^\alpha_{kki} = 0, \ \forall i, \alpha. \tag{2.18}$$

We will need the following generalized *Maximum Principle* due to Omori and Yau (cf. [18] and [23]).

Lemma 2.1. *Let M^n be a complete Riemannian manifold with Ricci curvature bounded from below and let $F : M^n \to \mathbb{R}$ be a C^2-function which is bounded from below on M^n. Then there is a sequence of points $\{p_k\}$ in M^n such that* $\lim_{k\to\infty} F(p_k) = \inf(F)$, $\lim_{k\to\infty} \mid \nabla F(p_k) \mid = 0$ *and* $\liminf_{k\to\infty} \Delta F(p_k) \geq 0$.

We also will need the following algebraic Lemma (for a proof see [21]).

Lemma 2.2. *Let $A, B : \mathbb{R}^n \to \mathbb{R}^n$ be symmetric linear maps such that $AB - BA = 0$ and $trA = trB = 0$. Then*

$$\mid trA^2B \mid \leq \frac{n-2}{\sqrt{n(n-1)}} N(A)\sqrt{N(B)} \tag{2.19}$$

and the equality holds if and only if $n-1$ of the eigenvalues x_i of A and the corresponding eigenvalues y_i of B satisfy

$$\begin{aligned} &| x_i |= \sqrt{\frac{N(A)}{n(n-1)}}, \ \ x_i x_j \geq 0, \\ &y_i = \sqrt{\frac{N(B)}{n(n-1)}} \left(\text{resp.}\, y_i = -\sqrt{\frac{N(B)}{n(n-1)}} \right). \end{aligned} \tag{2.20}$$

3. Proof of Simons' type Inequality

Proof of Theorem 1.10. If $H \neq 0$, set $\Phi_{ij}^{\alpha} = h_{ij}^{\alpha} - H^{\alpha}\delta_{ij}$ and consider the following symmetric tensor

$$\Phi = \sum_{\alpha,i,j} \Phi_{ij}^{\alpha} \omega_i \omega_j e_{\alpha}. \tag{3.1}$$

It is easy to check that Φ is traceless and

$$\begin{aligned} &N(\Phi^{\alpha}) = N(h^{\alpha}) - n(H^{\alpha})^2; \\ &| \Phi |^2 = \sum_{\alpha} N(\Phi^{\alpha}) = S - nH^2, \end{aligned} \tag{3.2}$$

where Φ^{α} denotes the matrix $[\Phi_{ij}^{\alpha}]$.

Because h is parallel, we have H constant. Moreover, as $H \neq 0$, we can choose a local field of orthonormal frames $\{e_1, e_2, \cdots, e_{n+p}\}$ such that $e_{n+1} = \dfrac{h}{H}$. With this choice (2.16) implies that

$$\begin{aligned} &h^{n+1}h^{\alpha} = h^{\alpha}h^{n+1}, \\ &\Phi_{ij}^{n+1} = h_{ij}^{n+1} - H\delta_{ij}, \\ &N(\Phi^{n+1}) = tr(h^{n+1})^2 - nH^2 = N(h^{n+1}) - nH^2, \\ &tr(h^{n+1})^3 = tr(\Phi^{n+1})^3 + 3HN(\Phi^{n+1}) + nH^3. \end{aligned} \tag{3.3}$$

$$\Phi_{ij}^{\alpha} = h_{ij}^{\alpha}, \ N(\Phi^{\alpha}) = N(h^{\alpha}), \ \alpha \geq n+2. \tag{3.4}$$

Since h is parallel, from (2.17), (3.2), (3.3) and (3.4) we have

$$\frac{1}{2}\Delta \mid \Phi \mid^2 = \frac{1}{2}\Delta S \geq n(1-H^2) \mid \Phi \mid^2 \ - nH \sum_{\alpha} tr(\Phi^{n+1}(\Phi^{\alpha})^2) + \sum_{\alpha,\beta} (tr \Phi^{\alpha}\Phi^{\beta})^2. \tag{3.5}$$

As the matrices Φ^{α} and Φ^{n+1} are traceless and the matrix Φ^{n+1} comutes with all the matrices Φ^{α}, we can apply Lemma 2.2 in order to obtain

$$\begin{aligned} \sum_{\alpha} \mathrm{tr}(\Phi^{n+1}(\Phi^{\alpha})^2) \ &\leq \frac{n-2}{\sqrt{n(n-1)}} \sqrt{N(\Phi^{n+1})} \mid \Phi \mid^2 \\ &\leq \frac{n-2}{\sqrt{n(n-1)}} \mid \Phi \mid^3 . \end{aligned} \tag{3.6}$$

Due to *Cauchy-Schwarz inequality* we can write

$$| \Phi |^4 \leq p \sum_{\alpha} N^2(\Phi^\alpha) \leq p \sum_{\alpha,\beta} (\mathrm{tr}\Phi^\alpha\Phi^\beta)^2. \tag{3.7}$$

It follows from (3.5), (3.6) and (3.7) that formula (1.5) holds.
If $H \equiv 0$, M^n is said to be maximal. In this case, from (1.2) we have

$$\frac{1}{2}\Delta S \geq S\left(\frac{S}{p} + n\right). \tag{3.8}$$

□

4. Proofs of Theorems 1.11 and 1.12

Proof of Theorem 1.11. Since the mean curvature vector h is parallel and $\sum_{\alpha,\beta,i,j,k} h^\alpha_{ij} h^\beta_{ki} R_{\alpha\beta jk} = \frac{1}{2}\sum_{\alpha,\beta} N(h^\alpha h^\beta - h^\beta h^\alpha)$, from (2.15) we have

$$\begin{aligned}\frac{1}{2}\Delta S &= \frac{1}{2}\sum_{\alpha,i,j}\Delta(h^\alpha_{ij})^2 = \sum_{\alpha,i,j,k}(h^\alpha_{ijk})^2 + \sum_{\alpha,i,j} h^\alpha_{ij}\Delta h^\alpha_{ij}\\ &= \sum_{\alpha,i,j,k}(h^\alpha_{ijk})^2 + \frac{1}{2}\sum_{\alpha,\beta} N(h^\alpha h^\beta - h^\beta h^\alpha)\\ &+ \sum_{\alpha,i,j,k,m} h^\alpha_{ij} h^\alpha_{km} R_{mijk} + \sum_{\alpha,i,j,k,m} h^\alpha_{ij} h^\alpha_{mi} R_{mkjk}.\end{aligned} \tag{4.1}$$

Next, we will obtain a pointwise estimate for the last two terms. For each fixed α, let λ^α_i be an eigenvalue of h^α, i.e. $h^\alpha_{ij} = \lambda^\alpha_i \delta_{ij}$, and denote by $\inf K$ the infimum of the sectional curvatures at a point p of M^n. Then

$$\begin{aligned}&2\left(\sum_{i,j,k,m} h^\alpha_{ij} h^\alpha_{km} R_{mijk} + \sum_{i,j,k,m} h^\alpha_{ij} h^\alpha_{mi} R_{mkjk}\right) =\\ &\sum_{i,k}(-2\lambda^\alpha_i\lambda^\alpha_k) R_{ikik} + \sum_{i,k}\left((\lambda^\alpha_i)^2 + (\lambda^\alpha_k)^2\right) R_{ikik} =\\ &\sum_{i,k}(\lambda^\alpha_i - \lambda^\alpha_k)^2 R_{ikik} \geq (\inf K)\sum_{i,k}(\lambda^\alpha_i - \lambda^\alpha_k)^2 =\\ &(\inf K)(2nN(h^\alpha) - 2n^2(H^\alpha)^2) = 2n(\inf K)N(\Phi^\alpha).\end{aligned} \tag{4.2}$$

It implies that

$$\begin{aligned}&\sum_{\alpha,i,j,k,m} h^\alpha_{ij} h^\alpha_{km} R_{mijk} + \sum_{\alpha,i,j,k,m} h^\alpha_{ij} h^\alpha_{mi} R_{mkjk} \geq\\ &n(\inf K)\sum_{\alpha} N(\Phi^\alpha) = n(\inf K) \mid \Phi \mid^2 .\end{aligned} \tag{4.3}$$

As h parallel implies H constant, by (2.7) we see that $S = R + n^2H^2 - n(n-1)$ is also constant, thus $\Delta S = 0$.

Since $R_{ijij} \geq 0$, from (4.1) and (4.3), we get

$$\begin{aligned} 0 = \frac{1}{2}\Delta S \geq \sum_{\alpha,i,j,k} (h^{\alpha}_{ijk})^2 + n(\inf K) \mid \Phi \mid^2 \\ + \frac{1}{2}\sum_{\alpha,\beta} N(h^{\alpha}h^{\beta} - h^{\beta}h^{\alpha}) \geq 0. \end{aligned} \tag{4.4}$$

It turns out that:
i) $h^{\alpha}h^{\beta} = h^{\beta}h^{\alpha}$, for all α and β and so the normal bundle of M^n is flat. Hence, all the matrices h^{α} can be diagonalized simultaneously;
ii) $h^{\alpha}_{ijk} = 0, \forall i,j,k,\alpha$ and so the second fundamental form B is parallel. In particular, it implies that λ^{α}_i is constant for all i, α.

From i), ii), (4.1) and (4.2) we can write $0 = \sum_{\alpha,i,j} (\lambda^{\alpha}_i - \lambda^{\alpha}_j)^2 R_{ijij}$ and, since $R_{ijij} \geq 0$, we obtain $(\lambda^{\alpha}_i - \lambda^{\alpha}_j)R_{ijij} = 0$.

Consequentely, we may apply the same methods used by Ishihara (see [12], Lemmas 5.1, 5.2 and Theorem 1.3) to conclude that M^n is totally umbilical or a product $M_1 \times M_2 \cdots \times M_k$, where M_i is a totally umbilical submanifold in $\mathbb{S}^{n+p}_p$ and the $M_i's$ are mutually perpendicular along their intersections. □

Remark: Let M^n be a complete spacelike submanifold in $\mathbb{S}^{n+p}_p(c)$ with parallel mean curvature vector and non-negative sectional curvature. In (4.4), we got the inequality $\Delta S \geq 0$, which shows that S is a subharmonic smooth function. Therefore, if the supremum of S is attained on M^n, it follows from the *Maximum Principle* that S is constant and we have the same conclusions as in Theorem 1.11.

Proof of Theorem 1.12. In the proof of Theorem 1.10 we used the following inequality

$$\begin{aligned} &\sum_{\alpha,i,j,k,m} h^{\alpha}_{ij}h^{\alpha}_{km}R_{mijk} + \sum_{\alpha,i,j,k,m} h^{\alpha}_{ij}h^{\alpha}_{mi}R_{mkjk} = \\ &n \mid \Phi \mid^2 - nH \sum_{\alpha} \operatorname{tr}\left(h^{n+1}(h^{\alpha})^2\right) + \sum_{\alpha,\beta} (\operatorname{tr}(h^{\alpha}h^{\beta}))^2 + \\ &\frac{1}{2}\sum_{\alpha,\beta} N(h^{\alpha}h^{\beta} - h^{\beta}h^{\alpha}) \geq \\ &\mid \Phi \mid^2 \left(\frac{\mid \Phi \mid^2}{p} - \frac{n(n-2)}{\sqrt{n(n-1)}} H \mid \Phi \mid + n(1-H^2) \right). \end{aligned} \tag{4.5}$$

Applying the same arguments as in the proof of the inequality (4.3), we obtain

$$\begin{aligned} &\sum_{\alpha,i,j,k,m} h^{\alpha}_{ij}h^{\alpha}_{km}R_{mijk} + \sum_{\alpha,i,j,k,m} h^{\alpha}_{ij}h^{\alpha}_{mi}R_{mkjk} \leq \\ &n \sup K \sum_{\alpha} N(\Phi^{\alpha}) = n \sup K \mid \Phi \mid^2 . \end{aligned} \tag{4.6}$$

For technical reasons, we will write the expression (4.1) for the Laplacian of S as

$$\begin{aligned}\frac{1}{2}\Delta \mid \Phi \mid^2 \geq (1-a)\left(\sum_{\alpha,i,j,k,m} h_{ij}^{\alpha}h_{km}^{\alpha}R_{mijk} + \sum_{\alpha,i,j,k,m} h_{ij}^{\alpha}h_{mi}^{\alpha}R_{mkjk}\right)\\ +a\left(\sum_{\alpha,i,j,k,m} h_{ij}^{\alpha}h_{km}^{\alpha}R_{mijk} + \sum_{\alpha,i,j,k,m} h_{ij}^{\alpha}h_{mi}^{\alpha}R_{mkjk}\right).\end{aligned} \tag{4.7}$$

Thus, from (4.5), (4.6) and (4.7), if $a \geq 1$, we have

$$\begin{aligned}\frac{1}{2}\Delta \mid \Phi \mid^2 \quad &\geq a \mid \Phi \mid^2 \left(\frac{\mid \Phi \mid^2}{p} - \frac{n(n-2)}{\sqrt{n(n-1)}} H \mid \Phi \mid \right.\\ &\left. + n[1 - H^2 + \left(\frac{1-a}{a}\right)\sup K]\right).\end{aligned} \tag{4.8}$$

Using similar arguments as in [14], it is possible to show that $\mid \Phi \mid^2 < \infty$. Therefore, we can apply Lemma 2.1 to the function $\mid \Phi \mid^2$ and obtain a sequence of points $\{p_k\}$ in M^n such that

$$\begin{aligned}&\lim_{k\to\infty} \mid \Phi \mid^2 (p_k) = \sup \mid \Phi \mid^2 = (\sup \mid \Phi \mid)^2,\\ &\lim_{k\to\infty} \mid \nabla \mid \Phi \mid^2 (p_k) \mid = 0 \text{ and } \limsup_{k\to\infty} \Delta \mid \Phi \mid^2 (p_k) \leq 0.\end{aligned} \tag{4.9}$$

By applying inequality (4.8) at p_k, taking the limit, and using (4.9) we get

$$\begin{aligned}0 \quad &\geq \frac{1}{2a}\limsup_{k\to\infty} \Delta \mid \Phi \mid^2 \geq (\sup \mid \Phi \mid)^2 \left(\frac{\sup \mid \Phi \mid^2}{p}\right.\\ &\left. - \frac{n(n-2)}{\sqrt{n\,(n-1)}} H \sup \mid \Phi \mid + n[1 - H^2 + \left(\frac{1-a}{a}\right)\sup K]\right).\end{aligned} \tag{4.10}$$

If $\sup K \leq \beta(n,p,H) = \dfrac{a}{4(a-1)(n-1)}\left(4(n-1) - [p(n-2)^2 + 4(n-1)]H^2\right)$, it can be easily checked that

$$\left(\frac{(\sup \mid \Phi \mid)^2}{p} - \frac{n(n-2)}{\sqrt{n\,(n-1)}} H \sup \mid \Phi \mid + n[1 - H^2 + \left(\frac{1-a}{a}\right)\sup K]\right) \geq 0,$$

and the equality holds if and only if $\sup K = \beta(n,p,H)$ and $\sup \mid \Phi \mid = \dfrac{pn(n-2)}{2\sqrt{n\,(n-1)}}$.

Thus, if $\sup K < \beta(n,p,H)$, from (4.10) and the last inequality we conclude that $\sup \mid \Phi \mid = 0$ and M^n is totally umbilical.

If $\sup K = \beta(n,p,H)$, we will suppose that M^n is not totally umbilical and derive a contradiction. First, let us prove that $p = 1$. Notice that

$$(\sup \mid \Phi \mid)^2 \left(\frac{(\sup \mid \Phi \mid)^2}{p} - \frac{n(n-2)}{\sqrt{n\,(n-1)}} H \sup \mid \Phi \mid + n[1 - H^2 + \left(\frac{1-a}{a}\right)\sup K]\right) = 0.$$

It shows that all the estimates used to obtain inequality (4.10) turn into equalities. More precisely, (3.6) and (3.7) can now be written as

$$\sqrt{N(\Phi^{n+1})} \mid \Phi \mid^2 = \mid \Phi \mid^3 . \tag{4.11}$$

$$\mid \Phi \mid^4 = p \sum_\alpha N^2(\Phi^\alpha). \tag{4.12}$$

As mentioned before, taking subsequences if necessary, we can arrive to a sequence $\{p_k\}$ in M^n, which satisfies (4.9) and such that

$$\lim_{k\to\infty} N(\Phi^\alpha)(p_k) = C^\alpha, \ \alpha \geq n+1. \tag{4.13}$$

By evaluating (4.11) at p_k, taking the limit for $k \to \infty$ and using (4.13) it gives

$$\sqrt{C^{n+1}} \left(\sup \mid \Phi \mid\right)^2 = \sup \mid \Phi \mid^3 = \left(\sup \mid \Phi \mid\right)^3 , \tag{4.14}$$

Since $\sup \mid \Phi \mid > 0$, we have

$$C^{n+1} = (\sup \mid \Phi \mid)^2 = \sup(\mid \Phi \mid^2) = \sum_\alpha C^\alpha. \tag{4.15}$$

Hence, $C^\alpha = 0, \forall \alpha \geq n+2$. By evaluating (4.12) at p_k and taking the limit for $k \to \infty$, from (4.13) and (4.15), we get

$$(\sup \mid \Phi \mid)^4 = p \sum_\alpha (C^\alpha)^2 = p(C^{n+1})^2 = p(\sup \mid \Phi \mid)^4,$$

which implies $p = 1$.

Next, let us prove that $\sup K = 0$. Since h is parallel and the equality holds in (4.6) and (4.7), we arrive to

$$0 = \limsup_{k\to\infty} \frac{1}{2} \Delta \mid \Phi \mid^2 (p_k) = \limsup_{k\to\infty} \frac{1}{2} \Delta S(p_k) = n(\sup K) \sup \mid \Phi \mid^2 = n(\sup K)(\sup \mid \Phi \mid)^2.$$

Therefore, $\sup K = 0$.

Now we are in position to prove that M^n is totally umbilical. Observe that $\sup K = 0$ and $p = 1$ yield

$$0 = \sup K = \beta(n,1,H) = \frac{a}{4(a-1)(n-1)} \left(4(n-1) - n^2 H^2\right).$$

Hence $H^2 = \frac{4(n-1)}{n^2}$. In this case, according to Montiel (cf. [16], Proposition 2), either M^n is a totally umbilical hypersurface or $n > 2$ and the supremum of the scalar curvature of M^n is equal to $(n-2)^2$.

As M^n is not totally umbilical, we conclude that the supremum of the scalar curvature of M^n is equal to $(n-2)^2$, which contradicts the fact that $\sup K = 0$. Therefore, M^n is totally umbilical.

Because a is arbitrary, taking the limit for $a \to \infty$ in $\sup K \leq \beta(n,p,H) = \frac{a}{4(a-1)(n-1)} \left(4(n-1) - [p(n-2)^2 + 4(n-1)]H^2\right)$, we get $\sup K \leq \beta(n,p,H) = \frac{1}{4(n-1)} \left(4(n-1) - [p(n-2)^2 + 4(n-1)]H^2\right)$.

Moreover, since M^n is totally umbilical, if $n \geq 3$ we obtain
$1 - H^2 = \sup K \leq \frac{1}{4(n-1)}\left(4(n-1) - [p(n-2)^2 + 4(n-1)]H^2\right)$, thus
$p(n-2)H^2 \leq 0$, which implies $H = 0$ and shows that M^n is totally geodesic. □

Acknowledgements. The authors would like to express their thanks to Fernanda Ester C. Camargo for valuable comments and suggestions about this paper, as well as to the referee for his careful reading of the original manuscript. This work was carried out while the second author was visiting the Institute of Mathematics and Statistics at the University of São Paulo (Brazil). He would like to thank Professor Claudio Gorodski and Professor Paolo Piccione for the warm hospitality and financial support, during his visit.

References

[1] R. Aiyama, *Compact spacelike m-submanifolds in a pseudo-Riemannian sphere* $\mathbb{S}_p^{m+p}(c)$, Tokyo J. Math. 18 no 1 (1995), 81-90.

[2] K. Akutagawa, *On spacelike hypersurfaces with constant mean curvature in the de Sitter space*, Math. Z. 196 (1987), 13 - 19.

[3] L.J. Alias and A. Romero, *Integral formulas for compact spacelike n-submanifolds in De Sitter spaces. Applications to the parallel mean curvature vector case*, Manuscripta Math. 87(1995), 405-416.

[4] A. Brasil, R.M.B. Chaves and G. Colares, *Rigidity results for submanifolds with parallel mean curvature vector in De Sitter space*, Glasg. Math. J. 48 (2006), 1-10.

[5] A. Brasil, G. Colares and O. Palmas, *Complete spacelike hypersurfaces with constant mean curvature in the de Sitter space: A gap theorem*, Illinois J. of Math, 47 (2003).

[6] E. Calabi, *Examples of Bernstein problems for some nonlinear eqquations*, Math. Proc. Cambridge Phil. Soc. 82 (1977), 489-495.

[7] Q.M. Cheng , *Complete space-like submanifolds in de Sitter space with parallel mean curvature vector*, Math. Z. 206 (1991), 333-339.

[8] S.Y. Cheng and S.T. Yau, *Maximal spacelike hypersurfaces in the Lorentz-Minkowski spaces*, Ann. of Math.(2) 104 (1976), 407-419.

[9] S.S. Chern, M. do Carmo and S. Kobayashi, *Minimal submanifolds of the sphere with second fundamental form of constant length*, Functional Analysis and Related Fields (F. Browder, ed.), Springer-Verlag, Berlin, (1970).

[10] M. Dajczer and K. Nomizu, *On the flat surfaces in* $\mathbb{S}_1^3$ *and* $\mathbb{H}_1^3$, Manifolds and lie Groups Birkauser, Boston, (1981).

[11] A.J. Goddard, *Some remarks on the existence of spacelike hypersurfaces of constant mean curvature*, Math. Proc. Cambridge Phil. Soc. 82 (1977), 489-495.

[12] T. Ishihara, *Maximal spacelike submanifolds of a pseudo-Riemannian space of constant curvature*, Mich. Math. J. 35 (1988), 345-352.

[13] U.H. Ki, H.J. Kim and H. Nakagawa, *On spacelike hypersurfaces with constant mean curvature of a Lorentz space form*, Tokyo J.Math. 14 (1991), 205-216.

[14] H. Li, *Complete spacelike submanifolds in de Sitter space with parallel mean curvature vector satisfying* $H^2 = \frac{4(n-1)}{n^2}$, Annals of Global Anal. and Geom. 15 (1997), 335-345.

[15] S. Montiel, *An integral inequality for compact spacelike hypersurfaces in the de Sitter space and applications to case of constant mean curvature*, Indiana Univ. Math. J. 37 (1988), 909-917.

[16] S. Montiel, *A characterization of hyperbolic cylinders in the de Sitter space*, Tohoku Math. J. 48 (1996), 23-31.

[17] S. Nishikawa, *On spacelike hypersurfaces in a Lorentzian manifold*, Nagoya Math. J. 95 (1984), 117-124.

[18] H. Omori, *Isometric Immersions of Riemannian manifolds*, J. Math. Soc. Japan 19 (1967), 205-214.

[19] J. Ramanathan, *Complete spacelike hypersurfaces of constant mean curvature in de Sitter space*, Indiana University Math. J. 36 (1987), 349-359.

[20] J. Simons, *Minimal varieties in Riemannian manifolds*, Ann. of Math. 88 (1968), 62-105.

[21] W. Santos, *Submanifolds with parallel mean curvature vector in spheres*, Tohoku Math. J. 46 (1994), 403-415.

[22] S.T. Yau, *Submanifolds with constant mean curvature II*, Amer. J. Math. 97 (1975), 78-100.

[23] S.T. Yau, *Harmonic functions on complete Riemannian manifolds*, Comm. Pure Appl. Math. 28 (1975), 201-228.

[24] K. Yano and S. Ishihara, *Submanifolds with parallel mean curvature vector*, J. Diff. Geom. 6 (1971), 95-118.

Rosa Maria S. Barreiro Chaves
Instituto de Matemática e Estatística
Universidade de São Paulo, Rua do Matão, 1010,
São Paulo - SP, Brazil, CEP 05508-090
rosab@ime.usp.br

Luiz Amancio M. Sousa Jr.
Departamento de Matemática e Estatística
Universidade Federal do Estado do Rio de Janeiro, Avenida Pasteur, 458,
Urca, Rio de Janeiro - RJ, Brazil, CEP 22290-240
amancio@impa.br

Recibido: 17 de noviembre de 2005
Aceptado: 22 de septiembre de 2006