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QUATERNIONS AND OCTONIONS IN MECHANICS

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1. Introduction

This is a survey of some of the ways in which Quaternions, Octonions and the exceptional group G_2 appear in today's Mechanics, addressed to a general audience.

The ultimate reason of this appearance is that quaternionic multiplication turns the 3-sphere of unit quaternions into a group, acting by rotations of the 3-space of purely imaginary quaternions, by

$$v \mapsto qvq^{-1}.$$

In fact, this group is Spin(3), the 2-fold cover of $SO(3)$, the group of rotations of $\mathbb{R}^3$.

This has been known for quite some time and is perhaps the simplest realization of Hamilton's expectations about the potential of quaternions for physics. One reason for the renewed interest is the fact that the resulting substitution of matrices by quaternions speeds up considerably the numerical calculation of the composition of rotations, their square roots, and other standard operations that must be performed when controlling anything from aircrafts to robots: four cartesian coordinates beat three Euler angles in such tasks.

A more interesting application of the quaternionic formalism is to the motion of two spheres rolling on each other without slipping, i.e., with infinite friction, which we will discuss here. The possible trajectories describe a vector 2-distribution on the 5-fold $S^2 \times S^3$, which depends on the ratio of the radii and is completely non-integrable unless this ratio is 1. As pointed out by R. Bryant, they are the same as those studied in Cartan's famous 5-variables paper, and contain the following surprise: *for all ratios different from 1:3 (and 1:1), the symmetry group is $SO(4)$, of dimension 6; when the ratio is 1:3 however, the group is a 14-dimensional exceptional simple Lie group of type G_2.*

The quaternions $\mathbb{H}$ and (split) octonions $\mathbb{O}_s$ help to make this evident, through the inclusion

$$S^2 \times S^3 \hookrightarrow \Im(\mathbb{H}) \times \mathbb{H} = \Im(\mathbb{O}_s).$$

The distributions themselves can be described in terms of pairs of quaternions, a description that becomes "algebraic over $\mathbb{O}_s$" in the 1:3 case. As a consequence, $Aut(\mathbb{O}_s)$, which is preciely that exceptional group, acts by symmetries of the system.

This phenomenon has been variously described as "the 1:3 rolling mystery", "a mere curiosity", "uncanny" and "the first appearance of an exceptional group in real life". Be as it may, it is the subject of current research and speculation. For the history and recent mathematical developments of rolling systems, see [Agrachev][Bor-Montgomery][Bryant-Hsu][Zelenko].

The technological applications deserve a paragraph, given that this Volume is dedicated to the memory of somebody especially preoccupied with the misuse of beautiful scientific discoveries. Quaternions are used to control the flight of aircrafts due to the advantages already cited, and "aircrafts" include guided missiles. A look at the most recent literature reveals that research in the area is being driven largely with the latter in mind. Octonions and G_2, on the other hand, although present in Physics via Joyce manifolds, seem to have had no technological applications so far – neither good nor bad. Still, the main application of Rolling Systems is to Robotics, a field with plenty to offer, of both kinds. The late Misha was rather pesimistic about the chances of the good eventually outweighting the bad. "Given the current state of the world", he said about a year before his death, "the advance of technology appears to be more dangerous than ever".

I would like to thank Andrei Agrachev for introducing me to the subject; John Baez, Gil Bor, Robert Bryant, Robert Montgomery and Igor Zelenko for enlightening exchanges; and the ICTP, for the fruitful and pleasant stay during which I became aquainted with Rolling Systems.

2. Quaternions and Rotations

Recall the quaternions,

$$\mathbb{H} = \{\mathbf{a} = a_o + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} : \ a_j \in \mathbb{R}\} \cong \mathbb{R}^4$$

as a real vector space, endowed with the bilinear multiplication $\mathbf{ab}$ defined by the relations

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1 \qquad \mathbf{ij} = \mathbf{k} = -\mathbf{ji} \qquad \mathbf{jk} = \mathbf{i} = -\mathbf{kj} \qquad \mathbf{ki} = \mathbf{j} = -\mathbf{ik}.$$

$\mathbb{H}$ is an associative algebra, like $\mathbb{R}$ or $\mathbb{C}$, where every non-zero element has an inverse, satisfying $\mathbf{a}^{-1}(\mathbf{ab}) = \mathbf{b} = (\mathbf{ba})\mathbf{a}^{-1}$, i.e., it is a division algebra. But unlike $\mathbb{R}$ or $\mathbb{C}$, it is clearly not commutative.

$\mathbb{H}$ can also be defined as pairs of complex numbers – much as $\mathbb{C}$ consists of pairs of real numbers. One sets

$$\mathbb{H} = \mathbb{C} \times \mathbb{C}$$

with product

$$(A, B)(C, D) = (AC - D\bar{B}, \bar{A}D + CB).$$

Under the equivalence, $\mathbf{i} = (0, 1)$, $\mathbf{j} = (i, 0)$, $\mathbf{k} = (0, i)$, and the *conjugation*

$$\overline{a_o + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}} = a_o - a_1\mathbf{i} - a_2\mathbf{j} - a_3\mathbf{k}$$

becomes

$$\overline{(A, B)} = (\bar{A}, -B).$$

The formula $\mathbf{a}\overline{\mathbf{b}} = (a_o + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k})(b_o - b_1\mathbf{i} - b_2\mathbf{j} - b_3\mathbf{k}) = (a_ob_o + a_1b_1 + a_2b_2 + a_3b_3) + (...)\mathbf{i} + (...)\mathbf{j} + (...)\mathbf{k}$ shows that, just as in the case of $\mathbb{R}^2 = \mathbb{C}$, the euclidean inner product in $\mathbb{R}^4 = \mathbb{H}$ and the corresponding norm are

$$< \mathbf{a}, \mathbf{b} >= \Re(\mathbf{a}\overline{\mathbf{b}}), \qquad |\mathbf{a}|^2 = \mathbf{a}\bar{\mathbf{a}}.$$

Since $|\mathbf{ab}| = |\mathbf{a}||\mathbf{b}|$,

$$\mathbf{a}^{-1} = \frac{\overline{\mathbf{a}}}{|\mathbf{a}|^2}.$$

The role of quaternions in mechanics comes through identifying euclidean 3-space with the imaginary quaternions $\Im(\mathbb{H})$ (= span of $\mathbf{i}, \mathbf{j}, \mathbf{k}$) and the following fact: *under quaternionic multiplication, the unit 3-sphere*

$$S^3 = \{\mathbf{a} \in \mathbb{H} : \ |\mathbf{a}| = 1\}$$

is a group, and the map

$$S^3 \times \Im(\mathbb{H}) \to \Im(\mathbb{H}), \qquad (\mathbf{a}, \mathbf{b}) \mapsto \mathbf{aba}^{-1}$$

is an action of this group by rotations of 3-space.

Inded, multiplying by a unit quaternion $\mathbf{a}$ on the left or on the right, is a linear isometry of $\mathbb{H} \cong \mathbb{R}^4$, as well as conjugating by it

$$\rho_{\mathbf{a}}(b) = \mathbf{aba}^{-1} = \mathbf{ab}\bar{\mathbf{a}}.$$

The transformation $\rho_{\mathbf{a}}$ preserves $\Im(\mathbb{H})$, since for an imaginary $\mathbf{b}$, $\overline{\rho_{\mathbf{a}}(b)} = \overline{\mathbf{ab}\bar{\mathbf{a}}} = \bar{\bar{\mathbf{a}}}\bar{\mathbf{b}}\bar{\mathbf{a}} = \mathbf{a}\bar{\mathbf{b}}\bar{\mathbf{a}} = -\mathbf{ab}\bar{\mathbf{a}} = -\rho_{\mathbf{a}}(b)$. In fact, $\rho_{\mathbf{a}}$ is isometry of $\Im(\mathbb{H}) = \mathbb{R}^3$, i.e., an element of the orthogonal group $O(3)$. Indeed, $\rho(S^3) = SO(3)$, because S^3 is compact and connected, and

$$\rho : S^3 \to SO(3)$$

is a Lie group homomorphism. This is a 2-1 map:

$$\rho_{\mathbf{a}} = \rho_{-\mathbf{a}}.$$

In fact,

$$\mathbb{Z}_2 \hookrightarrow S^3 \to SO(3)$$

is the universal cover of $SO(3)$. In particular, the fundamental group of the rotation group is

$$\pi_1(SO(3)) = \mathbb{Z}_2.$$

This “topological anomaly” of 3-space has been noted for a long time, and used too: if it wasn’t for it, there could be no rotating bodies – wheels, centrifuges, or turbines – fed by pipes or wires connected to the outside. In practice, by turning the latter twice for every turn of the body, the resulting “double twist” can be undone by translations.

Quaternions themselves come in when fast computation of composition of rotations, or square roots thereof, are needed, as in the control of an aircraft. For this, one needs coordinates for the rotations – three of them, since SO(3) is the group de matrices

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad A^tA = I, \qquad \det A = 1$$

and 9 parameters minus 6 equations leave 3 free parameters.

To coordinatize $SO(3)$ one uses the Euler angles, or variations thereof, of a rotation, obtained by writing it as a product $E_\alpha F_\beta E_\gamma$ where

$$E_\alpha = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad F_\alpha = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{bmatrix}.$$

But in

$$(E_{\alpha_1}F_{\beta_1}E_{\gamma_1})(E_{\alpha_2}F_{\beta_2}E_{\gamma_2}) = E_{\alpha_3}F_{\beta_3}E_{\gamma_3}$$

$$(E_{\alpha_1}F_{\beta_1}E_{\gamma_1})^{1/2} = E_{\alpha_3}F_{\beta_3}E_{\gamma_3}$$

the funcions $\alpha_3, \beta_3, \gamma_3$ are complicated expressions in $\alpha_1, \beta_1, \gamma_1, \alpha_2, \beta_2, \gamma_2$. Furthermore, when large rotations are involved, the multivaluedness and singularities of the Euler angles also lead to what numerical programmers know as "computational glitch". Instead, S^3 is easier to coordinatize, the formula for the quaternionic product is quadratic, and for $|\mathbf{a}| = 1$, $\mathbf{a} \neq -1$,

$$\sqrt{\mathbf{a}} = \frac{1}{\sqrt{2(\Re(\mathbf{a}) + 1)}} \left(\mathbf{a} + 1\right)$$

The price paid by these simplifications is the need of the non-linear condition $|\mathbf{a}|^2 = 1$. There is an extensive recent literature assesing the relative computational advantages of each representation, easily found in the web.

3. Rolling spheres

The configuration space of a pair of adjacent spheres is $S^2 \times SO(3)$. Indeed, we can assume one of the spheres Σ_1 to be the unit sphere $S^2 \subset \mathbb{R}^3$. Then, the position of the other sphere Σ_r is given by the point of contact $\mathbf{q} \in S^2$, together with an oriented orthonormal frame F attached to Σ_r. This may be better visualized by substituting momentarely Σ_r by an aircraft moving over the Earth Σ_1 at a constant height, a system whose configuration space is the same (airplane pilots call the frame F the "attitude" of the plane). Identifying F with the rotation $\rho \in SO(3)$ such that $\rho(F_o) = F$, where F_o is the standard frame in $\mathbb{R}^3$, the configuration is then given by the pair

$$(\mathbf{q}, \rho) \in S^2 \times SO(3).$$

Now let Σ_r roll on Σ_1 describing the curve $(\mathbf{q}(t), \rho(t)) \in S^2 \times SO(3)$. The non-slipping condition is encoded into two equations, expressing the vanishing of the linear and of the angular components of the slipping ("no slipping or twisting"), namely

$$(NS) \qquad (1+\frac{1}{r})\ \mathbf{q}' = \rho'\rho^{-1}(\mathbf{q})$$

$$(NT) \qquad \omega \perp \mathbf{q}$$

where $\omega \times v = \rho'\rho^{-1}(v)$ is the angular velocity of $\Sigma_r(t)$ relative to the fixed frame F_o. (NS) says that the linear velocity of the point of contact on the fixed Σ_1 is the same as the velocity of the point of contact on $\Sigma_r(t)$:

$$\mathbf{q}'(t) = \rho(t)\big((\rho(t)^{-1}(-r\mathbf{q}(t)))'\big)$$

The right-hand side is just the formula for transforming between rotating frames, given that the point of contact on $\Sigma_r(t)$ relative to the fixed frame F_o is $-r\mathbf{q}(t)$ plus a translation. Explicitely, relative to the frame $F_r(t)$ this point is (dropping the t's) $-r\rho^{-1}(\mathbf{q})$ and moves with velocity

$$(\rho^{-1}(-r\mathbf{q}))' = r\rho^{-1}\rho'\rho^{-1}\mathbf{q} - r\rho^{-1}\mathbf{q}'.$$

When rotated back to its actual position in $\mathbb{R}^3$, i.e., relative to the frame F_o, it becomes

$$\rho\big((\rho^{-1}(-r\mathbf{q}))'\big) = r\rho'\rho^{-1}\mathbf{q} - r\mathbf{q}'.$$

which is the same as $\mathbf{q}' = r\rho'\rho^{-1}\mathbf{q} - r\mathbf{q}'$, or

$$\frac{1+r}{r}\mathbf{q}' = \rho'\rho^{-1}(\mathbf{q}),$$

as claimed. (NT) is clearer, stating that Σ_r can rotate only about the axis perpendicular to the direction of motion and, because of (NS), tangent to Σ_1.

4. Rolling with quaternions

From now on, we will abandon the use of boldface letters for quaternions.

Replace the configuration space $S^2 \times SO(3)$ by its 2-fold cover $S^2 \times S^3$, viewed quaternionically as

$$S^2 \times S^3 \hookrightarrow \Im(\mathbb{H}) \times \mathbb{H},$$

and recall the map $Q \to \rho_Q$ from S^3 to $SO(3)$, $\rho_Q(v) = QvQ^{-1}$. Clearly,

$$T_{(q_o,Q_o)}(S^2 \times S^3) \cong \{(p,P) \in \Im(\mathbb{H}) \times \mathbb{H} : <p,q_o>=0=<P,Q_o>\}$$

Theorem. *A rolling trajectory $(q(t),\rho(t)) \in S^2 \times SO(3)$ satisfies (NS) and (NT) if and only if $\rho(t) = \rho_{q(t)Q(t)}$, where $(q(t),Q(t)) \in S^2 \times S^3$ is tangent to the distribution*

$$D^{(r)}_{(q_o,Q_o)} = \{(q_o x, \frac{1-r}{2r} x Q_o) :\ x \in q_o^{\perp} \subset \Im(\mathbb{H})\}.$$

Proof: $(p(t), Q(t))$ is tangent to $D^{(r)}$ if and only if for some smooth $x = x(t) \in p(t)^{\perp} \cap \Im(H)$, $p' = px$ and $Q' = \frac{1-r}{2r} xQ$. Eliminating x,

$$(*) \qquad \frac{1-r}{2r} p' = pQ'Q^{-1}$$

For a fixed v,

$$\rho(t)(v) = (p(t)Q(t))v(p(t)Q(t))^{-1} = -pQv\bar{Q}p,$$

and $\rho^{-1}(v) = -\bar{Q}pvpQ$, $\rho'(v) = -(p'Q + pQ')v\bar{Q}p - pQv(\bar{Q}'p + \bar{Q}p')$, and therefore $\rho'(\rho^{-1}(v)) = -p'pv - pQ'\bar{Q}pv - vpQ\bar{Q}'p - vpp'$. In particular,

$$\rho'(\rho^{-1}(p)) = 2p' + pQ'\bar{Q} + Q\bar{Q}'p$$

The (NS)-condition for $(x(t), \rho(t))$ is then

$$\frac{1+r}{r} p' = \rho'\rho^{-1}(p) = 2p' + pQ'\bar{Q} + Q\bar{Q}'p.$$

Since $x \perp p$, so is $Q'Q^{-1} = Q'\bar{Q}$ and, because p is purely imaginary, $pQ'\bar{Q} = Q\bar{Q}'p$. We conclude that the last equation is the same as $\frac{1-r}{r} p' = -2pQ'\bar{Q}$, as claimed. The rest of the proof proceeds along the same lines.

The distribution $D = D^{(r)}$ is integrable if and only if $r = 1$, that is, the spheres have the same radius. Otherwise, it is completely non integrable, of type (2,3,5), meaning that vector fields lying in it satisfy $\dim\{X + [Y, Z]\} = 3$, $\dim\{X + [Y, Z] + [U, [V, W]]\} = 5$. These are the subject of E. Cartan's famous "Five Variables paper" and were recognized as rolling systems by R. Bryant. Cartan and Engel provided the first realization of the exceptional group G_2 as the group of automorphisms of this differential system for $r = 3, 1/3$, the connection with "Cayley octaves" being made only later.

5. Symmetries

Given a vector distribution D on a manifold M, a global symmetry of it is a diffeomorphism of M that carries D to itself. They form a group, $Sym(M, D)$. But most often one needs local difeomorphisms too, hence the object of interest is really the Lie algebra $sym(M, D)$, but we shall not emphasize the distinction until it becomes significant.

If D is integrable, $sym(M, D)$ is infinite-dimensional, as can easily be seen by foliating the manifold. At the other end, if D is completely non-integrable ("bracket generating"), $sym(M, D)$ is generically trivial.

The rolling systems just described all have a $SO(4) = SO(3) \times SO(3)$ symmetry, as can be deduced from the physical set up. More formally, a pair of rotations (g_1, g_2) acts on $S^2 \times SO(3)$ by

$$(g_1, g_2) \cdot (x, \rho) = (g_1 x, g_1 \rho g_2^{-1})$$

This action preserves each of the $D^{(r)}$'s and are clearly global. Indeed, these are the only global symmetries that these distributions have for any $r \neq 1$.

In the covering space $S^2 \times S^3$, however, the action of $SO(4)$ extends to an action of a group of type G_2, yielding local diffeomorphisms of the configuration space, as we see next. More precisely,

$$S^2 \times S^3 = SO(4)/SO(2) = G_2'/P$$

where P is maximal parabolic. However, the lifted distributions $D^{(r)}$ themselves are not left invariant under the G_2-action – except in the case $r = 1/3$.

6. Octonions

The realization $\mathbb{H} = \mathbb{C} \times \mathbb{C}$ can be continued recursively to define the sequence of *Cayley-Dickson algebras*:

$$\mathbf{A}_{N+1} = \mathbf{A}_N \times \mathbf{A}_N$$

with product and conjugation

$$(A, B)(C, D) = (AC - D\bar{B}, \bar{A}D + CB) \qquad \overline{(A, B)} = (\bar{A}, -B).$$

Starting with $\mathbf{A}_0 = \mathbb{R}$,

$$\mathbf{A}_1 = \mathbb{C}, \qquad \mathbf{A}_2 = \mathbb{H}, \qquad \mathbf{A}_3 = \mathbb{O},$$

the algebra of *octonions*, which is non-associative. These four give essentially all division algebras $/\mathbb{R}$; from $\mathbf{A}_4$ – the *sedenions* – on, they have zero divisors, i.e., nonzero elements $\mathbf{a}, \mathbf{b}$ such that $\mathbf{ab} = 0$.

There is a *split* version of these algebras, where the product is obtained by changing the first minus in the formula by a +:

$$(A, B)(C, D) = (AC + D\bar{B}, \bar{A}D + CB).$$

Both the standard and the split versions can be expressed as direct sums

$$\mathbf{A}_{N+1} = \mathbf{A}_N \oplus \ell \mathbf{A}_N$$

where $\ell^2 = \pm 1$ according if it is the split one or not. Note that in a split algebra, $(1 + \ell)(1 - \ell = 0$, hence they have zero divisors from the start.

The main contribution of the Cayley-Dickson algebras to mathematics so far has been the fact that the automorphisms of the octonions provide the simplest realization of Lie groups of type G_2. More precisely, the complex Lie group of this type is the group of automorphisms of the complex octonions (i.e., with complex coefficients), its compact real form arises similarly from the ordinary real octonions and a non-compact real form G_2' arises from the split one. In physics, the Joyce manifolds of CFT carry, by definition, riemannian metrics with the compact G_2 as holonomy, while in rolling it is $G_2' = Aut(\mathbb{O}_s)$ that matters.

Since

$$\Im(\mathbb{O}_s) = \Im(\mathbb{H}) \times \mathbb{H} \cong \Im(\mathbb{H}) \oplus \ell\mathbb{H}$$

we can write

$$S^2 \times S^3 = \{\mathbf{a} = q + \ell Q \in \Im(\mathbb{O}_s) : \ |q| = 1 = |Q|\}.$$

These split octonions all have square zero: $(p+\ell Q)^2 = (pp+Q\bar{Q}) + \ell(-pQ+pQ)$. Indeed, every imaginary split octonion $\mathbf{a} \neq 0$ satisfying $\mathbf{a}^2 = 0$, is a positive multiple of one in $S^2 \times S^3$.

The formula for the product in $\mathbb{O}_s$ yields $\ell(xQ) = (\ell Q)x$ so that for all r the distributions can be written as

$$D^{(r)}_{(q+\ell Q)} = \{\mathbf{b} = \left(q + \left(\frac{1-r}{2r}\right)\ell Q\right) x \in \Im(\mathbb{O}_s) : \ x \in q^\perp \subset \Im(\mathbb{H})\}$$

In particular, $D^{1/3}_{\mathbf{a}} = \{\mathbf{a}x : \ x \in q^\perp \subset \Im(\mathbb{H})\}$. This expression is still not all octonionic, but its canonical extension to a 3-distribution $\tilde{D}_{\mathbf{a}} = D^{1/3}_{\mathbf{a}} + \mathbb{R}\mathbf{a}$ on the cone $\mathbb{R}_+(S^2 \times S^3)$ is:

<u>Lemma</u>: *For every octonion* $\mathbf{a} \in S^2 \times S^3$,

$$D^{1/3}_{\mathbf{a}} + \mathbb{R}\mathbf{a} = \{\mathbf{b} \in \Im(\mathbb{O}_s) : \ \mathbf{ab} = 0\},$$

a subspace we will denote by $Z_{\mathbf{a}}$.

To prove the Lemma, note that every subalgebra of a $\mathbb{O}_s$ generated by two elements is associative (i.e., $\mathbb{O}_s$ is "alternative"). Therefore $\mathbf{a}(\mathbf{a}x) = \mathbf{a}^2 x = 0$, proving one inclusion. The other uses the quadratic form associated to the split octonions, which also clarifies de action of $Aut(\mathbb{O}_s)$. It is $\Re(\mathbf{a}\overline{\mathbf{b}})$ which on $\Im(\mathbb{O}_s)) \cong \mathbb{R}^7$ can be replaced by its negative

$$< \mathbf{a}, \mathbf{b} >= \Re(\mathbf{ab}).$$

This is a symmetric and non-degenerate, of signature $(3,4)$ – in contrast to the one for ordinary Octonions, which is positive definite. Moreover, $\mathbf{a}^2 = 0 \Leftrightarrow < \mathbf{a}, \mathbf{a} >= 0$ for imaginary $\mathbf{a}$. It follows that $\mathbb{R}_+(S^2 \times S^3)$ is the null cone of the quadratic form, and the same as the set of elements of square zero in $\Im(\mathbb{O}_s)$. It is now easy to see that if $\mathbf{a}^2 = 0$ and $\mathbf{ab} = 0$, then $\mathbf{b} = \lambda\mathbf{a} + \mathbf{a}x$ with x as required.

Now, consider the group $G = Aut(\mathbb{O}_s)$, a non-compact simple Lie group of type G_2 and dimension 14. It fixes 1. On $\Im(\mathbb{O}_s)$, which is the orthogonal complement of 1 under $< \mathbf{a}, \mathbf{b} >$, this form is just $-\Re(\mathbf{ab})$, which is also preserved by G. Hence the quadratic form on all of $\mathbb{O}_s$ is G-invariant, hence so is $\Im(\mathbb{O}_s)$. This determines an inclusion

$$G \subset SO(3,4).$$

In particular, G acts linearly on the null cone of the form there. This action descends to a non-linear, transitive action on $S^2 \times S^3$ – much like the action of $SL(n,\mathbb{R})$ on $\mathbb{R}^n$ descends to one on S^{n-1}. Since $g(Z_{\mathbf{a}}) = Z_{g(\mathbf{a})}$, the action preserves the descended $Z_{\mathbf{a}}$'s, which are just the fibers of the distribution D. Hence

$$Aut(\mathbb{O}_s) \subset Aut(S^2 \times S^3, D)$$

In fact, the two sides are equal.

On the configuration space of the rolling system, the elements of G act only locally, via the local liftings of the covering map $S^2 \times S^3 \to S^2 \times SO(3)$. The local action, of course, still preserves the distribution $D^{1/3}$.

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