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SIMULTANEOUS APPROXIMATION BY A NEW SEQUENCE OF SZÃSZ–BETA TYPE OPERATORS

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ABSTRACT. In this paper, we study some direct results in simultaneous approximation for a new sequence of linear positive operators $M_n(f(t);x)$ of Szãsz–Beta type operators. First, we establish the basic pointwise convergence theorem and then proceed to discuss the Voronovaskaja-type asymptotic formula. Finally, we obtain an error estimate in terms of modulus of continuity of the function being approximated.

1. INTRODUCTION

In [3] Gupta and others studied some direct results in simultaneous approximation for the sequence:

$$B_n(f(t);x) = \sum_{k=0}^{\infty} q_{n,k}(x) \int_0^{\infty} b_{n,k}(t)\, f(t)\, dt,$$

where $x, t \in [0,\infty)$, $q_{n,k}(x) = \frac{e^{-nx}(nx)^k}{k!}$ and $b_{n,k}(t) = \frac{\Gamma(n+k+1)}{\Gamma(n)\Gamma(k+1)} t^k (1+t)^{-(n+k+1)}$. After that, Agrawal and Thamer [1] applied the technique of linear combination introduced by May [4] and Rathore [5] for the sequence $B_n(f(t);x)$. Recently, Gupta and Lupas [2] studied some direct results for a sequence of mixed Beta–Szãsz type operator defined as $L_n(f(t);x) = \sum_{k=1}^{\infty} b_{n,k}(x) \int_0^{\infty} q_{n,k-1}(t) f(t)\ dt + (1+x)^{-n-1} f(0)$.

In this paper, we introduce a new sequence of linear positive operators $M_n(f(t);x)$ of Szãsz–Beta type operators to approximate a function $f(x)$ belongs to the space $C_\alpha[0,\infty) = \{f \in C[0,\infty) : |f(t)| \le C(1+t)^\alpha \text{ for some } C > 0, \alpha > 0\}$, as follows:

$$M_n(f(t);x) = \sum_{k=1}^{\infty} q_{n,k}(x) \int_0^{\infty} b_{n,k-1}(t) f(t)\ dt + e^{-nx} f(0), \tag{1.1}$$

We may also write the operator (1.1) as $M_n(f(t);x) = \int_0^{\infty} W_n(t,x) f(t)\, dt$ where $W_n(t,x) = \sum_{k=1}^{\infty} q_{n,k}(x)\, b_{n,k-1}(t) + e^{-nx}\delta(t)$, $\delta(t)$ being the Dirac-delta function.

Key words and phrases. Linear positive operators, Simultaneous approximation, Voronovaskaja-type asymptotic formula, Degree of approximation, Modulus of continuity.

The space $C_\alpha\,[0,\infty)$ is normed by $\|f\|_{C_\alpha} = \sup\limits_{0\le t<\infty} |f(t)|\,(1+t)^{-\alpha}$.

There are many sequences of linear positive operators with are approximate the space $C_\alpha\,[0,\infty)$. All of them (in general) have the same order of approximation $O(n^{-1})$ [6]. So, to know what is the different between our sequence and the other sequences, we need to check that by using the computer. This object is outside our study in this paper.

Throughout this paper, we assume that C denotes a positive constant not necessarily the same at all occurrences, and $[\beta]$ denotes the integer part of β.

2. PRELIMINARY RESULTS

For $f \in C\,[0,\infty)$ the Szãsz operators are defined as $S_n(f;x) = \sum\limits_{k=1}^{\infty} q_{n,k}(x)\, f\left(\frac{k}{n}\right)$, $x \in [0,\infty)$ and for $m \in N^0$(the set of nonnegative integers), the m-th order moment of the Szãsz operators is defined as $\mu_{n,m}(x) = \sum\limits_{k=0}^{\infty} q_{n,k}(x)\left(\frac{k}{n}-x\right)^m$.

LEMMA 2.1. [3] *For $m \in N^0$, the function $\mu_{n,m}(x)$ defined above, has the following properties:*

(i) $\mu_{n,0}(x) = 1$, $\mu_{n,1}(x) = 0$, *and the recurrence relation is*

$$n\mu_{n,m+1}(x) = x\left(\mu'_{n,m}(x) + m\mu_{n,m-1}(x)\right),\ m \ge 1;$$

(ii) $\mu_{n,m}(x)$ *is a polynomial in x of degree at most $[m/2]$;*

(iii) *For every*$x \in [0,\infty)$,$\mu_{n,m}(x) = O\left(n^{-[(m+1)/2]}\right)$.

From above lemma, we get

$$\begin{aligned} \sum_{k=1}^{\infty} q_{n,k}(x)(k-nx)^{2j} &= n^{2j}\left(\mu_{n,2j}(x) - (-x)^{2j}e^{-nx}\right) \\ &= n^{2j}\left\{O(n^{-j}) + O(n^{-s})\right\} \quad \text{(for any } s>0) \\ &= O(n^j) \qquad\qquad\qquad\qquad \text{(if } s \ge j). \end{aligned} \tag{2.1}$$

For $m \in N^0$, the m-th order moment $T_{n,m}(x)$ for the operators (1.1) is defined as:

$$T_{n,m}(x) = M_n\left((t-x)^m;x\right) = \sum_{k=1}^{\infty} q_{n,k}(x)\int_0^{\infty} b_{n,k-1}(t)(t-x)^m dt + (-x)^m e^{-nx}.$$

LEMMA 2.2.*For the function $T_{n,m}(x)$, we have $T_{n,0}(x) = 1$, $T_{n,1}(x) = \frac{x}{n-1}$, $T_{n,2}(x) = \frac{nx^2+2nx+2x^2}{(n-1)(n-2)}$ and there holds the recurrence relation:*

$$\begin{aligned} (n-m-1)T_{n,m+1}(x) &= xT'_{n,m}(x) + \left((2x+1)m + x\right)T_{n,m}(x) \\ &\quad + mx(x+2)T_{n,m-1}(x), \quad n > m+1. \end{aligned} \tag{2.2}$$

Further, we have the following consequences of $T_{n,m}(x)$:

(i) $T_{n,m}(x)$ *is a polynomial in* x *of degree exactly* m*;*
(ii) *For every* $x \in [0,\infty)$, $T_{n,m}(x) = O\left(n^{-[(m+1)/2]}\right)$.

Proof: By direct computation, we have $T_{n,0}(x) = 1$, $T_{n,1}(x) = \frac{x}{n-1}$ and $T_{n,2}(x) = \frac{nx^2+2nx+2x^2}{(n-1)(n-2)}$. Next, we prove (2.2). For $x = 0$ it clearly holds. For $x \in (0,\infty)$, we have

$$T'_{n,m}(x) = \sum_{k=1}^{\infty} q'_{n,k}(x) \int_0^{\infty} b_{n,k-1}(t)(t-x)^m dt - n\,(-x)^m e^{-nx} - mT_{n,m-1}(x).$$

Using the relations $xq'_{n,k}(x) = (k-nx)\,q_{n,k}(x)$ and $t\,(1+t)\,b'_{n,k}(t) = (k-(n+1)\,t) \times b_{n,k}(t)$, we get:

$$\begin{aligned} xT'_{n,m}(x) &= \sum_{k=1}^{\infty} (k-nx)\,q_{n,k}(x) \int_0^{\infty} b_{n,k-1}(t)(t-x)^m dt + n(-x)^{m+1} e^{-nx} - mxT_{n,m-1}(x) \\ &= \sum_{k=1}^{\infty} q_{n,k}(x) \int_0^{\infty} t(1+t)\,b'_{n,k-1}(t)(t-x)^m dt + (n+1)T_{n,m+1}(x) - (-x)^{m+1} e^{-nx} \\ &\quad + (x+1)T_{n,m}(x) - (x+1)(-x)^m e^{-nx} - mxT_{n,m-1}(x). \end{aligned}$$

By using the identity $t(1+t) = (t-x)^2 + (1+2x)(t-x) + x(1+x)$, we have

$$\begin{aligned} xT'_{n,m}(x) &= \sum_{k=1}^{\infty} q_{n,k}(x) \int_0^{\infty} b'_{n,k-1}(t)\,(t-x)^{m+2} dt + (1+2x) \sum_{k=1}^{\infty} q_{n,k}(x) \int_0^{\infty} b'_{n,k-1}(t)(t-x)^{m+1} dt \\ &\quad + x(1+x) \sum_{k=1}^{\infty} q_{n,k}(x) \int_0^{\infty} b'_{n,k-1}(t)(t-x)^m dt + (n+1)T_{n,m+1}(x) \\ &\quad + (1+x)T_{n,m}(x) - mxT_{n,m-1}(x) - (-x)^m e^{-nx}. \end{aligned}$$

Integrating by parts, we get

$$xT'_{n,m}(x) = (n-m-1)\,T_{n,m+1}(x) - (m+x+2mx)\,T_{n,m}(x) - mx(x+2)\,T_{n,m-1}(x)$$

from which (2.2) is immediate.

From the values of $T_{n,0}(x)$ and $T_{n,1}(x)$, it is clear that the consequences (i) and (ii) hold for $m = 0$ and $m = 1$. By using (2.2) and the induction on m the proof of consequences (i) and (ii) follows, hence the details are omitted.

From the above lemma, we have

$$\begin{aligned} \sum_{k=1}^{\infty} q_{n,k}(x) \int_0^{\infty} b_{n,k-1}(t)(t-x)^{2r} dt &= T_{n,2r} - (-x)^{2r} e^{-nx} \qquad (2.3) \\ &= O(n^{-r}) + O(n^{-s}) \qquad \text{(for any } s > 0) \\ &= O(n^{-r}) \qquad \text{(if } s \geq r). \end{aligned}$$

LEMMA 2.3. *Let* δ *and* γ *be any two positive real numbers and* $[a,b] \subset (0,\infty)$. *Then, for any* $s > 0$, *we have*

$$\left\| \int_{|t-x|\geq\delta} W_n(t,x)\, t^{\gamma}\, dt \right\|_{C[a,b]} = O(n^{-s}).$$

Making use of Schwarz inequality for integration and then for summation and (2.3), the proof of the lemma easily follows.

LEMMA 2.4. [3] *There exist polynomials* $Q_{i,j,r}(x)$ *independent of* n *and* k *such that*

$$x^r D^r(q_{n,k}(x)) = \sum_{\substack{2i+j\leq r \\ i,j\geq 0}} n^i (k-nx)^j Q_{i,j,r}(x)\, q_{n,k}(x), \text{ where } D = \frac{d}{dx}.$$

3. MAIN RESULTS

Firstly, we show that the derivative $M_n^{(r)}(f(t);x)$ is an approximation process for $f^{(r)}(x)$, $r = 1, 2, \ldots$.

Theorem 3.1. *If* $r \in N$, $f \in C_\alpha[0,\infty)$ *for some* $\alpha > 0$ *and* $f^{(r)}$ *exists at a point* $x \in (0,\infty)$, *then*

$$\lim_{n\to\infty} M_n^{(r)}(f(t);x) = f^{(r)}(x). \tag{3.1}$$

Further, if $f^{(r)}$ *exists and is continuous on* $(a-\eta, b+\eta) \subset (0,\infty)$, $\eta > 0$, *then (3.1) holds uniformly in* $[a,b]$.

Proof: By Taylor's expansion of f, we have

$$f(t) = \sum_{i=0}^{r} \frac{f^{(i)}(x)}{i!}(t-x)^i + \varepsilon(t,x)(t-x)^r,$$

where, $\varepsilon(t,x) \to 0$ as $t \to x$. Hence

$$M_n^{(r)}(f(t);x) = \sum_{i=0}^{r} \frac{f^{(i)}(x)}{i!} \int_0^\infty W_n^{(r)}(t,x)(t-x)^i dt + \int_0^\infty W_n^{(r)}(t,x)\,\varepsilon(t,x)\,(t-x)^r dt$$
$$:= I_1 + I_2.$$

Now, using Lemma 2.2 we get that $M_n(t^m;x)$is a polynomial in x of degree exactly

, for all $m \in N^0$. Further , we can write it as:

$$M_n(t^m;x) = \frac{(n-m-1)!\; n^m}{(n-1)!} x^m + \frac{(n-m-1)!\; n^{m-1}}{(n-1)!} m(m-1)\; x^{m-1} + O(n^{-2}). \tag{3.2}$$

Therefore,

$$
\begin{aligned}
I_1 &= \sum_{i=0}^{r} \frac{f^{(i)}(x)}{i!} \sum_{j=0}^{i} \binom{i}{j} (-x)^{i-j} \int_0^\infty W_n^{(r)}(t,x)\, t^j\, dt \\
&= \frac{f^{(r)}(x)}{r!} \left(\frac{(n-r-1)!}{(n-1)\,!} n^r r! \right) = f^{(r)}(x) \left(\frac{(n-r-1)!}{(n-1)\,!} n^r \right) \to f^{(r)}(x) \quad \text{as} \quad n \to \infty.
\end{aligned}
$$

Next, making use of Lemma 2.4 we have

$$
\begin{aligned}
|I_2| \leq & \sum_{\substack{2i+j \leq r \\ i,j \geq 0}} n^i \frac{|Q_{i,j,r}(x)|}{x^r} \sum_{k=1}^{\infty} q_{n,k}(x)\ |k-nx|^j \int_0^\infty b_{n,k-1}(t)\ |\varepsilon(t,x)|\ |t-x|^r\, dt \\
& \qquad + (nx)^r e^{-nx}\, |\varepsilon(0,x)| \\
:= & \ I_3 + I_4.
\end{aligned}
$$

Since $\varepsilon(t,x) \to 0$ as $t \to x$, then for a given $\varepsilon > 0$, there exists a $\delta > 0$ such that $|\varepsilon(t,x)| < \varepsilon$, whenever $0 < |t-x| < \delta$. For $|t-x| \geq \delta$, there exists a constant $C > 0$ such that $|\varepsilon(t,x)(t-x)^r| \leq C\,|t-x|^\gamma$.

Now, since $\sup\limits_{\substack{2i+j \leq r \\ i,j \geq 0}} \frac{|Q_{i,j,r}(x)|}{x^r} := M(x) = C\ \forall\, x \in (0,\infty)$. Hence,

$$
\begin{aligned}
I_3 \leq C & \sum_{\substack{2i+j \leq r \\ i,j \geq 0}} n^i \sum_{k=1}^{\infty} q_{n,k}(x)\, |k-nx|^j \left(\int_{|t-x|<\delta} b_{n,k-1}(t)\, \varepsilon\, |t-x|^r\, dt \right. \\
& \qquad \left. + \int_{|t-x|\geq\delta} b_{n,k-1}(t)\ |t-x|^\gamma\, dt \right) \\
:= & \ I_5 + I_6.
\end{aligned}
$$

Now, applying Schwartz inequality for integration and then for summation, (2.1) and (2.3) we led to

$$I_5 \le \varepsilon C \sum_{\substack{2i+j\le r\\ i,j\ge 0}} n^i \sum_{k=1}^{\infty} q_{n,k}(x)\ |k-nx|^j \left(\int_0^\infty b_{n,k-1}(t)\,dt\right)^{1/2} \left(\int_0^\infty b_{n,k-1}(t)(t-x)^{2r}dt\right)^{1/2}$$

$$\left(\text{since} \int_0^\infty b_{n,k-1}(t)\ dt = 1\right)$$

$$\le \varepsilon C \sum_{\substack{2i+j\le r\\ i,j\ge 0}} n^i \left(\sum_{k=1}^{\infty} q_{n,k}(x)(k-nx)^{2j}\right)^{1/2} \left(\sum_{k=1}^{\infty} q_{n,k}(x)\int_0^\infty b_{n,k-1}(t)(t-x)^{2r}dt\right)^{1/2}$$

$$\le \varepsilon C\, O(n^{-r/2}) \sum_{\substack{2i+j\le r\\ i,j\ge 0}} n^i O(n^{j/2}) = \varepsilon\, O(1).$$

Again using Schwarz inequality for integration and then for summation, in view of (2.1) and Lemma 2.3, we have

$$I_6 \le C \sum_{\substack{2i+j\le r\\ i,j\ge 0}} n^i \sum_{k=1}^{\infty} q_{n,k}(x)\ |k-nx|^j \int_{|t-x|\ge\delta} b_{n,k-1}(t)\ |t-x|^{\gamma}\ dt$$

$$\le C \sum_{\substack{2i+j\le r\\ i,j\ge 0}} n^i \sum_{k=1}^{\infty} q_{n,k}(x)\ |k-nx|^j \left(\int_0^\infty b_{n,k-1}(t)\,dt\right)^{1/2} \left(\int_{|t-x|\ge\delta} b_{n,k-1}(t)(t-x)^{2\gamma}dt\right)^{1/2}$$

$$\le C \sum_{\substack{2i+j\le r\\ i,j\ge 0}} n^i \left(\sum_{k=1}^{\infty} q_{n,k}(x)(k-nx)^{2j}\right)^{1/2} \left(\sum_{k=1}^{\infty} q_{n,k}(x)\int_{|t-x|\ge\delta} b_{n,k-1}(t)(t-x)^{2\gamma}dt\right)^{1/2}$$

$$\le O(n^{-s}) \sum_{\substack{2i+j\le r\\ i,j\ge 0}} n^i O(n^{j/2}) \quad (\text{for any } s>0)$$

$$= O\left(n^{r/2-s}\right) = o(1) \qquad (\text{for } \ s > r/2).$$

Now, since $\varepsilon > 0$ is arbitrary, it follows that $I_3 = o(1)$. Also, $I_4 \to 0$ as $n \to \infty$ and hence $I_2 = o(1)$, combining the estimates of I_1 and I_2, we obtain (3.1).

To prove the uniformity assertion, it sufficient to remark that $\delta(\varepsilon)$ in above proof can be chosen to be independent of $x \in [a,b]$ and also that the other estimates holds uniformly in $[a,b]$.

Our next theorem is a Voronovaskaja-type asymptotic formula for the operators $M_n^{(r)}(f(t);x)$, $r = 1\,,\ 2\,,\ \ldots\,.$

THEOREM 3.2. *Let $f \in C_\alpha[0,\infty)$ for some $\alpha > 0$. If $f^{(r+2)}$ exists at a point $x \in (0,\infty)$, then*

$$\lim_{n\to 0} n\left(M_n^{(r)}(f(t);x) - f^{(r)}(x)\right) = \frac{r(r+1)}{2} f^{(r)}(x) + ((r+1)x + r) f^{(r+1)}(x) + \frac{1}{2} x(x+2)\, f^{(r+2)}(x). \tag{3.3}$$

Further, if $f^{(r+2)}$ exists and is continuous on the interval $(a-\eta, b+\eta) \subset (0,\infty)$, $\eta > 0$, then (3.3) holds uniformly on $[a,b]$.

Proof: By the Taylor's expansion of $f(t)$, we get

$$\begin{aligned} M_n^{(r)}(f(t);x) &= \sum_{i=0}^{r+2} \frac{f^{(i)}(x)}{i!} M_n^{(r)}\left((t-x)^i;x\right) + M_n^{(r)}\left(\varepsilon(t,x)(t-x)^{r+2};x\right) \\ &:= I_1 + I_2, \end{aligned}$$

where $\varepsilon(t,x) \to 0$as $t \to x$.
By Lemma 2.2 and (3.2), we have

$$\begin{aligned} I_1 &= \sum_{i=r}^{r+2} \frac{f^{(i)}(x)}{i!} \sum_{j=r}^{i} \binom{i}{j} (-x)^{i-j} M_n^{(r)}(t^j;x) \\ &= \frac{f^{(r)}(x)}{r!} M_n^{(r)}(t^r;x) + \frac{f^{(r+1)}(x)}{(r+1)!}\left((r+1)(-x) M_n^{(r)}(t^r;x) + M_n^{(r)}(t^{r+1};x)\right) \\ &+ \frac{f^{(r+2)}(x)}{(r+2)!}\left(\frac{(r+2)(r+1)}{2} x^2 M_n^{(r)}(t^r;x) + (r+2)(-x) M_n^{(r)}(t^{r+1};x) + M_n^{(r)}(t^{r+2};x)\right) \\ &= f^{(r)}(x)\left(\frac{(n-r-1)!n^r}{(n-1)!}\right) \\ &\quad + \frac{f^{(r+1)}(x)}{(r+1)}\left\{(r+1)(-x)\left(\frac{(n-r-1)\,!n^r}{(n-1)\,!} r\,!\right)\right. \\ &\quad\quad \left. + \left(\frac{(n-r-2)\,!n^{r+1}}{(n-1)\,!}(r+1)!x + \frac{(n-r-2)\,!n^r}{(n-1)\,!}(r+1) r\, r\,!\right)\right\} \\ &+ \frac{f^{(r+2)}(x)}{(r+2)!}\left\{\frac{(r+1)(r+2)}{2} x^2 \left(\frac{(n-r-1)\,!n^r}{(n-1)!} r\,!\right)\right. \\ &\quad + (r+2)(-x)\left(\left(\frac{(n-r-2)\,!n^{r+1}}{(n-1)\,!}\right)(r+1)\,!x + \left(\frac{(n-r-2)!n^r}{(n-1)\,!}\right)(r+1)\,!r\right) \\ &+ \left.\left(\frac{(n-r-3)\,!n^{r+2}}{(n-1)\,!} . \frac{(r+2)!}{2} x^2 + \frac{(n-r-3)\,!n^{r+1}}{(n-1)\,!}(r+2)(r+1)(r+1)\,!x\right)\right\} + O(n^{-2}). \end{aligned}$$

Hence in order to prove (3.3) it suffices to show that $nI_2 \to 0$ as $n \to \infty$, which follows on proceeding along the lines of proof of $I_2 \to 0$ as $n \to \infty$ in Theorem 3.1.
The uniformity assertion follows as in the proof of Theorem 3.1.
Finally, we present a theorem which gives as an estimate of the degree of approximation by $M_n^{(r)}(.;x)$ for smooth functions.

THEOREM 3.3. *Let $f \in C_\alpha[0,\infty)$ for some $\alpha > 0$ and $r \le q \le r+2$. If $f^{(q)}$ exists and is continuous on $(a-\eta, b+\eta) \subset (0,\infty)$, $\eta > 0$, then for sufficiently large n,*

$$\left\| M_n^{(r)}(f(t);x) - f^{(r)}(x) \right\|_{C[a,b]} \le C_1 n^{-1} \sum_{i=r}^{q} \left\| f^{(i)} \right\|_{C[a,b]} + C_2\, n^{-1/2} \omega_{f^{(q)}}\left(n^{-1/2}\right) + O(n^{-2})$$

where C_1, C_2 are constants independent of f and n, $\omega_f(\delta)$ is the modulus of continuity of f on $(a-\eta, b+\eta)$, and $\|\,.\,\|_{C[a,b]}$ denotes the sup-norm on $[a,b]$.

Proof. By Taylor's expansion of f, we have

$$f(t) = \sum_{i=0}^{q} \frac{f^{(i)}(x)}{i!}(t-x)^i + \frac{f^{(q)}(\xi) - f^{(q)}(x)}{q!}(t-x)^q \chi(t) + h(t,x)(1-\chi(t)),$$

where ξ lies between t, x, and $\chi(t)$ is the characteristic function of the interval $(a-\eta, b+\eta)$. Now,

$$M_n^{(r)}(f(t);x) - f^{(r)}(x) = \left(\sum_{i=0}^{q} \frac{f^{(i)}(x)}{i!} \int_0^\infty W_n^{(r)}(t,x)(t-x)^i dt - f^{(r)}(x) \right)$$

$$+ \int_0^\infty W_n^{(r)}(t,x) \left\{ \frac{f^{(q)}(\xi) - f^{(q)}(x)}{q!}(t-x)^q \chi(t) \right\} dt + \int_0^\infty W_n^{(r)}(t,x) h(t,x)(1-\chi(t))\, dt$$

$$:= I_1 + I_2 + I_3.$$

By using Lemma 2.2 and (3.2), we get

$$I_1 = \sum_{i=r}^{q} \frac{f^{(i)}(x)}{i!} \sum_{j=r}^{i} \binom{i}{j} (-x)^{i-j} \frac{d^r}{dx^r} \left(\frac{(n-j-1)!\; n^j}{(n-1)!} x^j \right.$$

$$\left. + \frac{(n-j-1)!\; n^{j-1}}{(n-1)!} j(j-1) x^{j-1} + O(n^{-2}) \right) - f^{(r)}(x).$$

Consequently,

$$\|I_1\|_{C[a,b]} \le C_1 n^{-1} \left(\sum_{i=r}^{q} \left\| f^{(i)} \right\|_{C[a,b]} \right) + O(n^{-2}), \text{uniformly on } [a,b].$$

To estimate I_2 we proceed as follows:

$$|I_2| \le \int_0^\infty \left| W_n^{(r)}(t,x) \right| \left\{ \frac{\left| f^{(q)}(\xi) - f^{(q)}(x) \right|}{q!} |t-x|^q \chi(t) \right\} dt$$

$$\le \frac{\omega_{f^{(q)}}(\delta)}{q!} \int_0^\infty \left| W_n^{(r)}(t,x) \right| \left(1 + \frac{|t-x|}{\delta} \right) |t-x|^q \, dt$$

$$\leq \frac{\omega_{f^{(q)}}(\delta)}{q!}\left[\sum_{k=1}^{\infty}\left|q_{n,k}^{(r)}(x)\right|\int_0^{\infty} b_{n,k-1}(t)\left(|t-x|^q+\delta^{-1}|t-x|^{q+1}\right)\,dt\right.$$
$$\left.+(-n)^r e^{-nx}(x^q+\delta^{-1}x^{q+1})\right],\ \delta>0.$$

Now, for $s = 0\,,1\,,\,2\,,\,\ldots\,,$ using Schwartz inequality for integration and then for summation, (2.1) and (2.3), we have

$$\sum_{k=1}^{\infty} q_{n,k}(x)\,|k-nx|^j \int_0^{\infty} b_{n,k-1}(t)\,|t-x|^s dt \leq \sum_{k=1}^{\infty} q_{n,k}(x)\,|k-nx|^j \left\{\left(\left(\int_0^{\infty} b_{n,k-1}(t)dt\right)\right)^{1/2}\right. \tag{3.4}$$
$$\left.\times\left(\int_0^{\infty} q_{n,k-1}(t)(t-x)^{2s}dt\right)^{1/2}\right\}$$
$$\leq \left(\sum_{k=1}^{\infty} q_{n,k}(x)(k-nx)^{2j}\right)^{1/2}\left(\sum_{k=1}^{\infty} q_{n,k}(x)\int_0^{\infty} b_{n,k-1}(t)(t-x)^{2s}dt\right)^{1/2}$$
$$= O(n^{j/2})O(n^{-s/2})$$
$$= O(n^{(j-s)/2}),\ \text{uniformly on}[a,b].$$

Therefore, by Lemma 2.4 and (3.4), we get

$$\sum_{k=1}^{\infty}\left|q_{n,k}^{(r)}(x)\right|\int_0^{\infty} b_{n,k-1}(t)\ |t-x|^s\ dt \leq \sum_{k=1}^{\infty}\ \sum_{\substack{2i+j\leq r\\ i,j\geq 0}} n^i\,|k-nx|^j\ \frac{|Q_{i,j,r}(x)\,|}{x^r}q_{n,k}(x) \tag{3.5}$$
$$\times\int_0^{\infty} b_{n,k-1}(t)\ \ |\,t-x|^s\ dt$$
$$\leq \left(\sup_{\substack{2i+j\leq r\\ i,j\geq 0}}\ \sup_{x\in[a,b]}\frac{|Q_{i,j,r}(x)|}{x^r}\right)\sum_{\substack{2i+j\leq r\\ i,j\geq 0}} n^i\left(\sum_{k=1}^{\infty} q_{n,k}(x)\,|k-nx|^j\int_0^{\infty} b_{n,k-1}(t)\ |t-x|^s\ dt\right)$$
$$= C\sum_{\substack{2i+j\leq r\\ i,j\geq 0}} n^i O(n^{(j-s)/2}) = O(n^{(r-s)/2}),\text{uniformly on } [a,b]\,.$$

(since $\sup_{\substack{2i+j\leq r\\ i,j\geq 0}}\ \sup_{x\in[a,b]}\frac{|Q_{i,j,r}(x)|}{x^r} := M(x)$ but fixed)

Choosing $\delta = n^{-1/2}$ and applying (3.5), we are led to

$$\|I_2\|_{C[a,b]} \leq \frac{\omega_{f^{(q)}}\left(n^{-1/2}\right)}{q!}\left[O(n^{(r-q)/2}) + n^{1/2}O(n^{(r-q-1)/2}) + O(n^{-m})\right],\text{(for any } m>0)$$
$$\leq C_2\,n^{-(r-q)/2}\omega_{f^{(q)}}\left(n^{-1/2}\right).$$

Since $t \in [0,\infty) \setminus (a-\eta, b+\eta)$, we can choose $\delta > 0$ in such a way that $|t-x| \geq \delta$ for all $x \in [a,b]$. Thus, by Lemmas 2.3 and 2.4 , we obtain

$$|I_3| \leq \sum_{k=1}^{\infty} \sum_{\substack{2i+j\leq r\\ i,j\geq 0}} n^i \left|k-nx\right|^j \frac{|Q_{i,j,r}(x)|}{x^r} q_{n,k}(x) \int_{|t-x|\geq\delta} b_{n,k-1}(t) \left|h(t,x)\right| \, dt + (-n)^r e^{-nx} \left|h(0,x)\right|.$$

For $|t-x| \geq \delta$, we can find a constant C such that $|h(t,x)| \leq C\left|t-x\right|^{\alpha}$. Hence, using Schwarz inequality for integration and then for summation ,(2.1), (2.3), it easily follows that $I_3 = O(n^{-s})$ for any $s > 0$, uniformly on $[a,b]$.

Combining the estimates of I_1 , I_2 I_3, the required result is immediate.

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