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AMALGAMATION PROPERTY IN QUASI-MODAL ALGEBRAS

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Abstract. In this paper we will give suitable notions of Amalgamation and Super-amalgamation properties for the class of quasi-modal algebras introduced by the author in his paper Quasi-Modal algebras.

1. Introduction and preliminaries

The class of quasi-modal algebras was introduced by the author in [2], as a generalization of the class of modal algebras. A quasi-modal algebra is a Boolean algebra A endowed with a map Δ that sends each element $a \in A$ to an ideal Δa of A, and satisfies analogous conditions to the modal operator $\Box$ of modal algebras [3]. This type of maps, called *quasi-modal operators*, are not operations on the Boolean algebra, but have some similar properties to modal operators.

It is known that some varieties of modal algebras have the Amalgamation Property (AP) and Superamalgamation Property (SAP). These properties are connected with the Interpolation property in modal logic (see [3]). The aim of this paper is to introduce a generalization of these notions for the class of quasi-modal algebras, topological quasi-modal algebras, and monadic quasi-modal algebras.

We recall some concepts needed for the representation for quasi-modal algebras. For more details see [2] and [1].

Let $A = \langle A, \vee, \wedge, \neg, 0, 1\rangle$ be a Boolean algebra. The set of all ultrafilters is denoted by $\mathrm{Ul}(A)$ .The ideal (filter) generated in A by some subset $X \subseteq A$, will be denoted by $I_A(X)$, $(F_A(X))$. The complement of a subset $Y \subseteq A$ will be denoted by Y^c or $A - Y$. The lattice of ideals (filters) of A is denoted by $Id(A)$ $(Fi(A))$.

Definition 1. Let A be a Boolean algebra. A *quasi-modal operator* defined on A is a function $\Delta : A \to Id(A)$ that verifies the following conditions for all $a, b \in A$:

Q1 $\Delta(a \wedge b) = \Delta a \cap \Delta b$,

Q2 $\Delta 1 = A$.

A *quasi-modal algebra*, or *qm*-algebra, is a pair $\langle A, \Delta\rangle$ where A is a Boolean algebra and Δ is a quasi-modal operator.

In every quasi-modal algebra we can define the dual operator $\nabla : A \to Fi(A)$ by $\nabla a = \neg\Delta\neg a$, where $\neg\Delta x = \{\neg y \mid y \in \Delta x\}$. It is easy to see that the operator ∇ satisfies the conditions

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(1) $\nabla(a \vee b) = \nabla a \cap \nabla b$, and
(2) $\nabla 0 = A$,

for all $a, b \in A$ (see [2]). The class of qm-algebras is denoted by $\mathcal{QMA}$.

Let A be a qm-algebra. For each $P \in \mathrm{Ul}(A)$ we define the set

$$\Delta^{-1}(P) = \{a \in A \mid \Delta a \cap P \neq \emptyset\}.$$

Lemma 2. [2] *Let $A \in \mathcal{QMA}$.*

(1) *For each $P \in \mathrm{Ul}(A)$, $\Delta^{-1}(P) \in Fi(A)$,*
(2) *$a \in \Delta^{-1}(P)$ iff for all $Q \in \mathrm{Ul}(A)$, if $\Delta^{-1}(P) \subseteq Q$ then $a \in Q$.*

Let $A \in \mathcal{QMA}$. We define on $\mathrm{Ul}(A)$ a binary relation R_A by

$$\begin{aligned}(P, Q) \in R_A \quad &\Leftrightarrow \quad \forall a \in A: \text{ if } \Delta a \cap P \neq \emptyset, \text{ then } a \in Q \\ &\Leftrightarrow \quad \Delta^{-1}(P) \subseteq Q.\end{aligned}$$

Throughout this paper, we will frequently work with Boolean subalgebras of a given Boolean algebra. In order to avoid any confusion, if $A \in \mathcal{QMA}$, then we will use the symbol Δ_A to denote the corresponding operation of A. A function $h : A \to B$ is an homomorphism of quasi-modal algebras, or a q-homomorphism, if h is an homomorphism of Boolean algebras, and

$$I_B(h(\Delta_A a)) = \Delta_B(h(a)),$$

for any $a \in A$. A *quasi-isomorphism* is a Boolean isomorphism that is a q-homomorphism.

Let $A \in \mathcal{QMA}$. Let us consider the relational structure

$$\mathcal{F}(A) = \langle \mathrm{Ul}(A), R_A \rangle.$$

From the result given in [2] it follows that the Boolean algebra $\mathcal{P}(\mathrm{Ul}(A))$ endowed with the operator

$$\overline{\Delta_{R_A}} : \mathcal{P}(\mathrm{Ul}(A)) \to Id(\mathcal{P}(\mathrm{Ul}(A)))$$

defined by

$$\overline{\Delta_{R_A}}(O) = I(\Delta_{R_A}(O)) = \{U \in \mathcal{P}(\mathrm{Ul}(A)) \mid U \subseteq \Delta_{R_A}(O)\},$$

is a quasi-modal algebra. Moreover, the map $\beta : A \to \mathcal{P}(\mathrm{Ul}(A))$ defined by $\beta(a) = \{P \in \mathrm{Ul}(A) \mid a \in P\}$, for each $a \in A$, is a q-homomorphism, i.e. $I_{\mathcal{P}(\mathrm{Ul}(A))}(\beta(\Delta a)) = \Delta_{R_A}(\beta(a))$, for each $a \in A$.

Let $A \in \mathcal{QMA}$. Let $X \subseteq A$. Define the ideal $\Delta X = I(\bigcup\limits_{x \in X} \Delta x)$ and the filter $\nabla X = F\left(\bigcup_{x \in X} \nabla x\right)$. For $a \in A$ we define recursively $\Delta^0 a = I(a)$, and $\Delta^{n+1} a = \Delta(\Delta^n a)$.

Theorem 3. [2] *Let $A \in \mathcal{QMA}$. Then for all $a \in A$, the following equivalences hold:*

1. *$\Delta a \subseteq I(a) \Leftrightarrow R_A$ is reflexive.*
2. *$\Delta a \subseteq \Delta^2 a \Leftrightarrow R_A^2 \subseteq R_A$, i.e., R_A is transitive.*
3. *$I(a) \subseteq \Delta \nabla a = \bigcap\limits_{x \in \nabla a} \Delta x \Leftrightarrow R_A$ is symmetrical.*

Let $A \in \mathcal{QMA}$. We shall say that A is a topological *quasi*-modal *algebra* if for every $a \in A$, $\Delta a \subseteq I(a)$ and $\Delta a \subseteq \Delta^2 a$. We shall say that A is a monadic *quasi*-modal *algebra* if it is a quasi-topological algebra and $I(a) \subseteq \Delta\nabla a$ for every $a \in A$. From the previous Theorem we get that a quasi-modal algebra A is quasi-topological algebra iff the relation R_A is reflexive and transitive. Similarly, a quasi-modal algebra A is a monadic quasi-modal algebra iff the relation R_A is an equivalence.

Let A and B be two qm-algebras. We shall say that the structure $B = \langle B, \vee, \wedge, \Delta_B, 0, 1\rangle$ is a *quasi-modal subalgebra* of A, or *qm-subalgebra* for short, if B is a Boolean subalgebra of A, and for any $a \in B$,

$$I_A(\Delta_B(a)) = \Delta_A(a).$$

The following result is given in [1] for Quasi-modal lattices.

Lemma 4. *Let A be a qm-algebra. Let B be a Boolean subalgebra of A. Then the following conditions are equivalent*

(1) *There is a quasi-modal operator, $\Delta_B : B \to Id(B)$, such that $B = \langle B, \vee, \wedge, \Delta_B, 0, 1\rangle$ is a qm-subalgebra of A.*
(2) *For any $a \in B$, $\Delta_B(a)$ is defined to be $I_A(\Delta_A(a) \cap B)$.*

Remark 5. Let A be a qm-algebra. Let B be a Boolean subalgebra of A. If there is a quasi-modal operator $\Delta_B : B \to Id(B)$, such that $B = \langle B, \vee, \wedge, \Delta_B, 0, 1\rangle$ is a qm-subalgebra of A, then Δ_B is unique. The proof of this fact is as follows: First, we note that if H is an ideal of B, and G is a filter of B such that $H \cap G = \emptyset$, then $I_A(H) \cap F_A(G) = \emptyset$. We suppose now that there exist two quasi-modal operators Δ_1 and Δ_2 in B such that $\langle B, \vee, \wedge, \Delta_1, 0, 1\rangle$ and $\langle B, \vee, \wedge, \Delta_2, 0, 1\rangle$ are two qm-subalgebras of A. Then

$$\Delta_A(a) = I_A(\Delta_1(a)) = I_A(\Delta_2(a)).$$

If $\Delta_1(a) \subsetneq \Delta_2(a)$, there exists $b \in B$ such that $b \in \Delta_1(a)$ and $b \notin \Delta_2(a)$. Then $\Delta_A(a) \cap F_A(b) = I_A(\Delta_1(a)) \cap F_A(b) \neq \emptyset$ and $\Delta_A(a) \cap F_A(b) = I_A(\Delta_2(a)) \cap F_A(b) = \emptyset$, which is a contradiction.

2. Amalgamation Property

It is known that the variety of modal algebras has the Amalgamation Property (AP) and the Superamalgamation Property (SAP) (see [3] for these properties and the connection with the Interpolation property in modal logic). In this section we shall give a generalization of these notions and prove that the class $\mathcal{QMA}$ has these properties.

Definition 6. Let $\mathcal{Q}$ be a class of quasi-modal algebras. We shall say that $\mathcal{Q}$ has the AP if for any triple A_0, $A_1, A_2 \in \mathcal{Q}$ and injective quasi-homomorphisms $f_1 : A_0 \to A_1$ and $f_2 : A_0 \to A_2$ there exists $B \in \mathcal{Q}$ and injective quasi-homomorphisms $g_1 : A_1 \to B$ and $g_2 : A_2 \to B$ such that $g_1 \circ f_1 = g_2 \circ f_2$, and $g_1(\Delta_{A_0} c) = g_2(\Delta_{A_0} c)$, for every $c \in A_0$.

We shall say that $\mathcal{Q}$ has the *SAP* if $\mathcal{Q}$ has the AP and in addition the maps g_1 and g_2 above have the following property:

For all $(a,b) \in A_1 \times A_2$ such that $g_1(a) \leq g_2(b)$ there exists $c \in A_0$ such that $a \leq f_1(c)$ and $f_2(c) \leq b$.

Let $\mathcal{Q}$ be a class of quasi-modal algebras. Without losing generality, we can assume that in the above definition A_0 is a qm-subalgebra of A_1 and A_2, i.e., that f_1 and f_2 are the inclusion maps.

Theorem 7. *The class $\mathcal{QMA}$ has the AP.*

Proof. Let $A_0, A_1, A_2 \in \mathcal{QMA}$ such A_0 is a qm-subalgebra of A_1 and A_2. Let us consider the relational structures $\mathcal{F}(A_i) = \langle \mathrm{Ul}(A_i), R_{A_i}, \beta(A_i) \rangle$, $1 \leq i \leq 2$. We shall define the set

$$X = \{(P,Q) \in \mathrm{Ul}(A_1) \times \mathrm{Ul}(A_2) \mid P \cap A_0 = Q \cap A_0\},$$

and the binary relation R in X as follows:

$$(P_1,Q_1)\, R\, (P_2,Q_2) \text{ iff } (P_1,P_2) \in R_{A_1} \text{ and } (Q_1,Q_2) \in R_{A_2}.$$

Let us consider the quasi-modal algebra $\mathcal{A}(X) = \left\langle \mathcal{P}(X), \cup, ^c, \bar{\Delta}_R, \emptyset \right\rangle$ where the operator

$$\bar{\Delta}_R : \mathcal{P}(X) \to Id(\mathcal{P}(X))$$

is defined by

$$\bar{\Delta}_R(G) = I(\Delta_R(G)) = \{U \in \mathcal{P}(X) \mid U \subseteq \Delta_R(G)\}.$$

We note that in this case $I(\Delta_R) = \Delta_R$.

Let us define the maps $g_1 : A_1 \to \mathcal{A}(X)$ and $g_2 : A_2 \to \mathcal{A}(X)$ by

$$g_1(a) = \{(P,Q) \in X : a \in P\} \text{ and } g_1(b) = \{(P,Q) \in X \mid b \in Q\},$$

respectively. We prove that g_1 and g_2 are injective q-homomorpshims.

First of all let us check the following property:

$$\forall P \in \mathrm{Ul}(A_1)\ \exists Q \in \mathrm{Ul}(A_2) \text{ such that } (P,Q) \in X. \tag{1}$$

Let $P \in \mathrm{Ul}(A_1)$. Since $P \cap A_0 \in \mathrm{Ul}(A_0)$ and A_0 is a subalgebra of A_2, we get by known results on Boolean algebras that there exists $Q \in \mathrm{Ul}(A_2)$ such that $P \cap A_0 = Q \cap A_0$, i.e., $(P,Q) \in X$.

To see that g_1 is injective, let $a,b \in A_1$ such that $a \nleq b$. Then there exists $P \in \mathrm{Ul}(A_1)$ such that $a \in P$ and $b \notin P$. By (1), there exists $Q \in \mathrm{Ul}(A_2)$ such that $(P,Q) \in X$. So, $(P,Q) \in g_1(a)$ and $(P,Q) \notin g_1(b)$, i.e., $g_1(a) \subsetneq g_1(b)$. Thus, g_1 is injective. It is clear that g_1 is a Boolean homomorphism.

We prove next that $\Delta_R(g_1(a)) \subseteq g_1(\Delta a)$, for any $a \in A_1$.

Let $(P,Q) \in X$. Suppose that $(P,Q) \notin g_1(\Delta_{A_1} a)$, i.e., $\Delta_{A_1} a \cap P = \emptyset$. From Lemma 2 there exists $P_1 \in \mathrm{Ul}(A_1)$ such that $(P,P_1) \in R_{A_1}$ and $a \notin P_1$. By property (1), there exists $Q_1 \in \mathrm{Ul}(A_2)$ such that $P_1 \cap A_0 = Q_1 \cap A_0$. From the inclusion $\Delta_{A_1}^{-1}(P) \subseteq P_1$, it is easy to see that $\Delta_{A_0}^{-1}(P \cap A_0) \subseteq P_1 \cap A_0$. Thus,

$$\Delta_{A_0}^{-1}(P \cap A_0) \subseteq Q_1 \cap A_0.$$

We prove that there exists $Q_2 \in \mathrm{Ul}(A_2)$ such that $P_1 \cap A_0 = Q_2 \cap A_0$ and $\Delta_{A_2}^{-1}(Q) \subseteq Q_2$. Let us consider the filter

$$F = F_{A_2}\left(\Delta_{A_2}^{-1}(Q) \cup (P_1 \cap A_0)\right).$$

The filter F is proper, because otherwise there exist $q \in \Delta_{A_2}^{-1}(Q)$ and $d \in P_1 \cap A_0$ such that $q \wedge d = 0$. So, $q \leq \neg d$, and since Δ_{A_2} is increasing, we get $\Delta_{A_2}(q) \subseteq \Delta_{A_2}(\neg d)$. As A_0 is a quasi-subalgebra of A_2 and $d \in A_0$, we get

$$\Delta_{A_0}(\neg d) = I_{A_2}(\Delta_{A_2}(\neg d) \cap A_0).$$

Thus,

$$I_{A_2}(\Delta_{A_2}(\neg d) \cap A_0) \cap Q \neq \emptyset.$$

So, there exists $x \in Q$ and $y \in \Delta_{A_2}(\neg d) \cap A_0$ such that $x \leq y$. Since $y \in Q \cap A_0 = P \cap A_0$, $y \in \Delta_{A_2}(\neg d) \cap A_0 \cap P$. Then we have $\neg d \in P_1$, which is a contradiction. Therefore, there exists $Q_2 \in \mathrm{Ul}(A_2)$ such that

$$P_1 \cap A_0 = Q_2 \cap A_0 \text{ and } \Delta_{A_2}^{-1}(Q) \subseteq Q_2.$$

So, $(P, Q) \notin \Delta_R g_1(a)$. So, we have the inclusion $\Delta_R g_1(a) \subseteq g_1(\Delta_{A_1} a)$. The inclusion $g_1(\Delta_{A_1} a) \subseteq \Delta_R g_1(a)$ is easy and left to the reader. Thus, $g_1(\Delta_{A_1} a) = \Delta_R g_1(a)$, and consequently we get

$$I(g_1(\Delta_{A_1} a)) = I(\Delta_R g_1(a)),$$

i.e., g_1 is an injective q-homomorphism.

Similarly we can prove that g_2 is an injective q-homomorphism. Moreover, it is easy to check that for all $c \in A_0$, $g_1(c) = g_2(c)$, and $g_1(\Delta_{A_0} c) = g_2(\Delta_{A_0} c)$. Thus, $\mathcal{QMA}$ has the AP. □

Lemma 8. *Let A_0, A_1 and $A_2 \in \mathcal{QMA}$ such that A_0 is a subalgebra of A_1 and A_2. Let $a \in A_1$, $b \in A_2$ and let us suppose that there exists no $c \in A_0$ such that $a \leq c$ or $c \leq b$. Then there exist $P \in \mathrm{Ul}(A_1)$, $Q \in \mathrm{Ul}(A_2)$ such that*

$$a \in P,\ b \notin Q \text{ and } P \cap A_0 = Q \cap A_0.$$

Proof. Let us consider the filter $F_{A_1}(a)$ in A_1 and the filter $F_{A_2}(F_{A_1}(a) \cap A_0)$ in A_2. We note that

$$b \notin F_{A_2}(F_{A_1}(a) \cap A_0),$$

because in otherwise there exists $c \in A_0$ such that $a \leq_1 c \leq_2 b$, which is a contradiction. Then by the Ultrafilter theorem, there exists $Q \in \mathrm{Ul}(A_2)$ that such that

$$F_{A_1}(a) \cap A_0 \subset F_{A_2}(F_{A_1}(a) \cap A_0) \subseteq Q \text{ and } b \notin Q.$$

Let us consider in A_1 the filter $F = F_{A_1}(\{a\} \cup (Q \cap A_0))$ and the ideal $I = I_{A_1}((A_2 - Q) \cap A_0)$. We prove that

$$F \cap I = \emptyset.$$

Suppose that there exist elements $x \in A_1$, $y \in Q \cap A_0$, and $z \in (A_2 - Q) \cap A_0$ such that $a \wedge y \leq_1 x \leq_1 z$. This implies that

$$a \leq_1 \neg y \vee z \in F_{A_1}(a) \cap A_0 \subseteq Q.$$

Thus, we get $z \in Q$, which is a contradiction. So, there exists $P \in \mathrm{Ul}(A_1)$ such that

$$a \in P, Q \cap A_0 = P \cap A_0 \text{ and } b \notin Q.$$

□

Theorem 9. *The class $\mathcal{QMA}$ has the SAP.*

Proof. The proof of the SAP is actually analogous to the previous one. Let A_0, $A_1, A_2 \in \mathcal{QMA}$ such that A_0 is a qm-subalgebra of A_1 and of A_2. Let us consider the set X and the quasi-modal algebra $\mathcal{A}(X)$ of the proof above.

Let $(a, b) \in A_1 \times A_2$. Suppose that there exists no $c \in A_0$ such that $a \leq c$ or $c \leq b$. By Lemma 8 there exists $P \in \mathrm{Ul}(A_1)$ and there exists $Q \in \mathrm{Ul}(A_2)$ such that

$$a \in P,\, b \notin Q \quad \text{and } P \cap A_0 = Q \cap A_0,$$

i.e., $(P, Q) \in g_1(a)$ and $(P, Q) \notin g_2(a)$. Thus, $g_1(a) \nsubseteq g_2(b)$. So, $\mathcal{QMA}$ has the SAP. □

The results above can be applied to prove that other classes of quasi-modal algebras have the AP and SAP. For example, if $\mathcal{TQMA}$ is the class of the topological quasi-modal algebras, then in the proof of Theorem 7 the binary relation R defined on the set X is reflexive and transitive. Consequently, $\mathcal{A}(X)$ is a topological quasi-modal algebra and thus the class $\mathcal{TQMA}$ has the AP and the SAP. Similar considerations can be applied to the class $\mathcal{MQMA}$ of monadic quasi-modal algebras.

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