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APPROXIMATION DEGREE FOR GENERALIZED INTEGRAL OPERATORS

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ABSTRACT. Very recently Jain et al. [4] proposed generalized integrated Baskakov operators $V_{n,\alpha}(f,x), \alpha > 0$ and estimated some approximation properties in simultaneous approximation. In the present paper we establish the rate of convergence of these operators and its Bezier variant, for functions which have derivatives of bounded variation.

1. Introduction

For $x \in [0,\infty)$ and $\alpha > 0$, the general family of Baskakov type operators considered in [3] is defined as

$$V_{n,\alpha}(f,x) = \sum_{v=1}^{\infty} p_{n,v,\alpha}(x) \int_0^\infty b_{n,v-1,\alpha}(t)f(t)dt + (1+\alpha x)^{-n/\alpha} f(0) \qquad (1)$$

where

$$p_{n,v,\alpha}(x) = \frac{\Gamma(n/\alpha + v)}{\Gamma(v+1)\Gamma(n/\alpha)} \cdot \frac{(\alpha x)^v}{(1+\alpha x)^{\frac{n}{\alpha}+v}}, b_{n,v,\alpha}(t) = \frac{\alpha\Gamma(n/\alpha+v+1)}{\Gamma(v+1)\Gamma(n/\alpha)} \cdot \frac{(\alpha t)^v}{(1+\alpha t)^{\frac{n}{\alpha}+v+1}}.$$

In the alternative form the above operators (1), can be defined as

$$V_{n,\alpha}(f(t),x) = \int_0^\infty W_{n,\alpha}(x,t)f(t)dt,$$

where the kernel $W_{n,\alpha}(x,t)$ in terms of Dirac delta function $\delta(t)$, is given by

$$W_{n,\alpha}(x,t) = \sum_{v=1}^{\infty} p_{n,v,\alpha}(x) b_{n,v-1,\alpha}(t) + (1+\alpha x)^{-n/\alpha}\delta(t).$$

We define $\beta_{n,\alpha}(x,t) = \int_0^t W_{n,\alpha}(x,s)ds$, then as a special case we have $\beta_{n,\alpha}(x,\infty) = \int_0^\infty W_{n,\alpha}(x,s)ds = 1$. Let $DB_\gamma(0,\infty), \gamma \geq 0$ be the class of absolutely continuous functions f defined on $(0,\infty)$ satisfying the growth condition $f(t) = O(t^\gamma), t \to \infty$ and having a derivative f' on the interval $(0,\infty)$ coinciding a.e. with a function which is of bounded variation on every finite subinterval of $(0,\infty)$. It can be observed that all functions $f \in BD_\gamma(0,\infty)$ posses for each $c > 0$ a representation

$$f(x) = f(c) + \int_c^x \psi(t)dt, \;\; x \geq c.$$

In [4] the authors studied some direct results in simultaneous approximation for the operators (1). Very recently the rate of convergence for bounded function for the operators $V_{n,\alpha}(f,x)$ has been obtained by Gangwar and Jain [3]. For certain operators Bojanic and Khan [2] and Taberska [10] estimated the rate of convergence for functions having derivative of B.V. The analogous problem on the convergence rate for the Bernstein polynomials and certain other integral operators were studied in [1],[5], [6] and [8]. Very recently Ispir et al. [9] considered the Kantorovich process of a generalized sequence of linear positive operators and estimated the rate of convergence for absolutely continuous functions having a derivative coinciding a.e., with a function of bounded variation.

As the operators defined by (1) are the generalized operators, this motivated us to extend the studies and here we study the approximation properties of the operators $V_{n,\alpha}(f,x)$ and its Bezier variant. We estimate the convergence rate for functions whose first derivative is of bounded variation.

2. Auxiliary Results

We shall use the following lemmas to prove our main theorem.

Lemma 1 [4]. *Let the $m-th$ order central moment be of the operators (1) be defined by*

$$T_{n,m,\alpha}(x)=\sum_{v=1}^{\infty}p_{n,v,\alpha}(x)\int_0^{\infty}b_{n,v-1,\alpha}(t)(t-x)^m dt+(1+\alpha x)^{-n/\alpha}(-x)^m.$$

Then, we have

$$T_{n,0,\alpha}(x)=1, T_{n,1,\alpha}=\frac{\alpha x}{n-\alpha},$$

and

$$T_{n,2,\alpha}(x)=\frac{x(1+\alpha x)(2n-\alpha)+\alpha x(1+3\alpha x)}{(n-\alpha)(n-2\alpha)}$$

and for $n>\alpha(m+1)$, we have

$$[n-\alpha(m+1)]T_{n,m+1,\alpha}(x)=x(1+\alpha x)\lfloor T^{(1)}_{n,m,\alpha}(x)+2mT_{n,m-1,\alpha}(x)\rfloor + [m(1+2\alpha x)+\alpha x]T_{n,m,\alpha}(x).$$

It is easily checked that for all $x\in[0,\infty)$, one has

$$T_{n,m,\alpha}(x)=O\big(n^{-[(m+1)/2]}\big).$$

Remark 1. In particular given any number $\lambda>2$ and $x>0$, by Lemma 1, we have for n sufficiently large one has

$$V_{n,\alpha}((t-x)^2,x)\equiv T_{n,2,\alpha}(x)\le\frac{\lambda x(1+\alpha x)}{n}. \tag{2}$$

Remark 2. In view of Remark 1, it can be observed by Holder's inequality that

$$V_{n,\alpha}(\mid t-x\mid,x)\le[T_{n,2,\alpha}(x)]^{1/2}\le\sqrt{\lambda x(1+\alpha x)/n}. \tag{3}$$

Lemma 2. *Let $x>0,\lambda>2$ and the kernel $W_{n,\alpha}(x,t)$ is defined by (1), then for n sufficiently large, we have*

$$(i)\ \eta_{n,\alpha}(x,y)=\int_0^y W_{n,\alpha}(x,t)dt\le\frac{\lambda x(1+\alpha x)}{n(x-y)^2},\ 0\le y<x$$

$$(ii)\ 1-\eta_{n,\alpha}(x,z)=\int_z^{\infty}W_{n,\alpha}(x,t)dt\leq\frac{\lambda x(1+\alpha x)}{n(z-x)^2},\ x<z<\infty$$

Proof. First we prove (i), by (2), we have

$$\int_0^y W_{n,\alpha}(x,t)dt\leq\int_0^y\frac{(x-t)^2}{(x-y)^2}W_{n,\alpha}(x,t)dt$$
$$\leq(x-y)^{-2}T_{n,2,\alpha}(x)\leq\frac{\lambda x(1+\alpha x)}{x(x-y)^2}.$$

The proof of (ii) is similar, we omit the details.

3. Main Result

In this section, we prove the following main theorem.

Theorem 1. *Let $f\in DB_\gamma(0,\infty)$,$\gamma>0$ and $x\in(0,\infty)$. Then for $\lambda>2$ and for n sufficiently large, we have*

$$\begin{aligned}|V_{n,\alpha}(f,x)-f(x)|\leq{}&\frac{\lambda(1+\alpha x)}{n}\Big(\sum_{v=1}^{[\sqrt{n}]}\bigvee_{x-x/v}^{x+x/v}((f')_x)+\frac{x}{\sqrt{n}}\bigvee_{x-x/\sqrt{n}}^{x+x/\sqrt{n}}((f')_x)\Big)\\&+\frac{\lambda(1+\alpha x)}{nx}(|f(2x)-f(x)-xf'(x^+)|+|f(x)|)\\&+\sqrt{\frac{\lambda x(1+\alpha x)}{n}}\Big(C2^\gamma O(n^{\frac{-\gamma}{2}})+|f'(x^+)|\Big)\\&+\frac{1}{2}\sqrt{\frac{\lambda x(1+\alpha x)}{n}}|f'(x^+)-f'(x^-)|\\&+\frac{\alpha x}{2(n-\alpha)}|f'(x^+)+f'(x^-)|,\end{aligned}$$

where $\bigvee_a^b f(x)$ denotes the total variation of f_x on $[a,b]$, and f_x is defined by

$$f_x(t)=\begin{cases}f(t)-f(x^-),\ 0\leq t<x;\\0,\quad t=x;\\f(t)-f(x^+),\ x<t<\infty.\end{cases}$$

Proof. Using the fact that $\int_0^\infty W_{n,\alpha}(x,t)dt=1$, we can write

$$V_{n,\alpha}(f,x)-f(x)=\int_0^\infty W_{n,\alpha}(x,t)(f(t)-f(x))dt$$
$$=\int_0^\infty\Big(\int_x^t W_{n,\alpha}(x,t)(f'(u)du)dt\Big).$$

Also, we can write

$$f'(u)=\frac{[f'(x^+)+f'(x^-)]}{2}+(f')_x(u)+\frac{[f'(x^+)-f'(x^-)]}{2}sgn(u-x)$$

$$+[f'(x)-\frac{[f'(x^+)+f'(x^-)]]}{2}\chi_x(u).$$

Next, we have $\int_0^\infty(\int_x^t f'(x)-\frac{[f'(x^+)+f'(x^-)]}{2}\chi_x(u)du))W_{n,\alpha}(x,t)dt=0$, thus

$$V_{n,\alpha}(f,x)-f(x)=\int_0^\infty(\int_x^t W_n(x,t)(\frac{[f'(x^+)+f'(x^-)]}{2}+(f')_x(u))du)dt$$

$$\int_0^\infty(\int_x^t W_n(x,t)\frac{[f'(x^+)-f'(x^-)]}{2}sgn(u-x)du)dt.$$

Also

$$\int_0^\infty(\int_x^t \frac{[f'(x^+)-f'(x^-)]}{2}sgn(u-x)du)W_{n,\alpha}(x,t)dt=\frac{[f'(x^+)-f'(x^-)]}{2}V_{n,\alpha}(\mid t-x\mid,x)$$

and

$$\int_0^\infty(\int_x^t \frac{1}{2}[f'(x^+)+f'(x^-)]du)W_{n,\alpha}(x,t)dt=\frac{1}{2}[f'(x^+)+f'(x^-)]V_{n,\alpha}((t-x),x).$$

Thus we can write

$$\begin{aligned}
&\mid V_{n,\alpha}(f,x)-f(x)\mid\\
&\le\mid\int_x^\infty(\int_x^t(f')_x(u)du)W_{n,\alpha}(x,t)dt-\int_0^x(\int_x^t(f')_x(u)du)W_{n,\alpha}(x,t)dt\mid\\
&\quad+\frac{1}{2}\mid f'(x^+)-f'(x^-)\mid V_{n,\alpha}(\mid t-x\mid,x)\\
&\quad+\frac{1}{2}\mid f'(x^+)+f'(x^-)\mid V_{n,\alpha}((t-x),x)\\
&=\mid A_{n,\alpha}(f,x)+B_{n,\alpha}(f,x)\mid+\frac{1}{2}\mid f'(x^+)-f'(x^-)\mid V_{n,\alpha}(\mid t-x\mid,x)\\
&\quad+\frac{1}{2}\mid f'(x^+)+f'(x^-)\mid V_{n,\alpha}((t-x),x). \qquad (4)
\end{aligned}$$

To complete the proof of the theorem it is sufficient to estimate the terms $A_n(f,x)$ and $B_n(f,x)$. Applying integration by parts, using Lemma 2 and taking $y=x-x/\sqrt{n}$,we have

$$\mid B_{n,\alpha}(f,x)\mid=\mid\int_0^x(\int_x^t(f')_x(u)du)d_t\eta_{n,\alpha}(x,t)dt\mid$$

$$\int_0^x\eta_{n,\alpha}(x,t)(f')_x(t)dt\le(\int_0^y+\int_y^x)\mid(f')_x(t)\mid\mid\eta_{n,\alpha}(x,t)\mid dt$$

$$\le\frac{\lambda x(1+\alpha x)}{n}\int_0^y\bigvee_t^x((f')_x)\frac{1}{(x-t)^2}dt+\int_y^x\bigvee_t^x((f')_x)dt$$

$$\le\frac{\lambda x(1+\alpha x)}{n}\int_0^y\bigvee_t^x((f')_x)\frac{1}{(x-t)^2}dt+\frac{x}{\sqrt{n}}\bigvee_{x-\frac{x}{\sqrt{n}}}^x((f')_x).$$

Let $u=\frac{x}{x-t}$. Then we have

$$\frac{\lambda x(1+\alpha x)}{n}\int_0^y \bigvee_t^x ((f')_x)\frac{1}{(x-t^2)}dt = \frac{\lambda x(1+\alpha x)}{n}\int_1^{\sqrt{n}} \bigvee_{x-\frac{x}{u}}^{x} ((f')_x)du$$

$$\leq \frac{\lambda(1+\alpha x)}{n}\sum_{v=1}^{[\sqrt{n}]}\bigvee_{x-\frac{x}{u}}^{x}((f')_x).$$

Thus

$$| B_{n,\alpha}(f,x) | \leq \frac{\lambda(1+\alpha x)}{n}\sum_{v=1}^{[\sqrt{n}]}\bigvee_{x-\frac{x}{u}}^{x}((f')_x) + \frac{x}{\sqrt{n}}\bigvee_{x-\frac{x}{\sqrt{n}}}^{x}((f')_x). \tag{5}$$

On the other hand, we have

$$\begin{aligned}
| A_{n,\alpha}(f,x) | =& | \int_x^\infty (\int_x^t (f')_x(u)du)W_{n,\alpha}(x,t)dt | \\
=& | \int_{2x}^\infty (\int_x^t (f')_x(u)du)W_{n,\alpha}(x,t)dt + \int_x^{2x}(\int_x^t (f')_x(u)du)dt(1-\eta_{n,\alpha}(x,t)) | dt \\
\leq& | \int_{2x}^\infty (f(t)-f(x))W_{n,\alpha}(x,t)dt | + | f'(x^+) || \int_{2x}^\infty (t-x)W_{n,\alpha}(x,t)dt | \\
&+ | \int_x^{2x}(f')_x(u)du) || (1-\eta_{n,\alpha}(x,2x) | + \int_x^{2x} | (f')_x(t) | | (1-\eta_{n,\alpha}(x,t) | dt \\
\leq& \frac{C}{x}\int_{2x}^\infty W_{n,\alpha}(x,t)t^\gamma | t-x | dt + \frac{| f(x) |}{x^2}\int_{2x}^\infty W_{n,\alpha}(x,t)(t-x)^2 dt \\
&+ | f'(x^+) | \int_{2x}^\infty W_{n,\alpha}(x,t) | (t-x) | dt + \frac{\lambda(1+\alpha x)}{nx}(| f(2x)-f(x)-xf'(x^+) | \\
&+ \frac{\lambda(1+\alpha x)}{n}\sum_{v=1}^{[\sqrt{n}]}\bigvee_x^{x+\frac{x}{v}}((f')_x) + \frac{x}{\sqrt{n}}\bigvee_x^{x+\frac{x}{\sqrt{n}}}((f')_x).
\end{aligned} \tag{6}$$

Next applying Holder's inequality, and Lemma 1, we proceed as follows for the estimation of the first two terms in the right hand side of (6):

$$\begin{aligned}
&\frac{C}{x}\int_{2x}^\infty W_{n,\alpha}(x,t)t^\gamma | t-x | dt + \frac{| f(x) |}{x^2}\int_{2x}^\infty W_{n,\alpha}(x,t)(t-x)^2 dt \\
&\leq \frac{C}{x}(\int_{2x}^\infty W_{n,\alpha}(x,t)t^{2\gamma}dt)^{\frac{1}{2}}(\int_0^\infty W_{n,\alpha}(x,t)(t-x)^2 dt)^{\frac{1}{2}} \\
&\qquad + \frac{| f(x) |}{x^2}(\int_{2x}^\infty W_{n,\alpha}(x,t)(t-x)^2 dt) \\
&\leq C2^\gamma O(n^{-\gamma/2})\frac{\sqrt{\lambda x(1+\alpha x)}}{\sqrt{n}} + | f(x) | \frac{\lambda(1+\alpha x)}{nx}.
\end{aligned} \tag{7}$$

Also the third term of the right side of (6) is estimated as

$$| f'(x^+) | \int_{2x}^\infty W_{n,\alpha}(x,t) | t-x | dt \leq | f'(x^+) | \int_0^\infty W_{n,\alpha}(x,t) | t-x | dt$$

$$\leq | f'(x^+) | (\int_0^\infty W_{n,\alpha}(x,t)(t-x)^2 dt)^{\frac{1}{2}} (\int_0^\infty W_{n,\alpha}(x,t)dt)^{\frac{1}{2}}$$

$$= | f'(x^+) | \frac{\sqrt{\lambda x(1+\alpha x)}}{\sqrt{n}}.$$

Collecting the estimates (4)-(7), we get the required result.
This completes the proof of Theorem 1.

4. Bezier Variant

For $\beta \geq 1$ or $0 < \beta < 1$, the Bezier variant of the operators (1) can be defined as

$$V_{n,\alpha,\beta}(f,x) = \sum_{v=1}^{\infty} Q_{n,v,\alpha}^{(\beta)}(x) \int_0^\infty b_{n,v-1,\alpha}(t) f(t) dt + Q_{n,0,\alpha}^{(\beta)}(x) f(0) \quad (8)$$

$$= \int_0^\infty W_{n,\alpha,\beta}(x,t) f(t) dt$$

where $Q_{n,v,\alpha}^{(\beta)}(x) = (J_{n,v,\alpha}(x))^\beta - (J_{n,v+1,\alpha}(x))^\beta$, $J_{n,v,\alpha}(x) = \sum_{j=v}^{\infty} p_{n,v,\alpha}(x)$ and the kernel is given by

$$W_{n,\alpha,\beta}(x,t) = \sum_{v=1}^{\infty} Q_{n,v,\alpha}^{(\beta)}(x) b_{n,v-1,\alpha}(t) + Q_{n,0,\alpha}^{(\beta)}(x)\delta(t),$$

$\delta(t)$ being Dirac delta function. In case $\beta = 1$ the operators defined by (8) reduce to the operators (1).
We define

$$\eta_{n,\alpha,\beta}(x,y) = \int_0^y W_{n,\alpha,\beta}(x,t)dt$$

Remark 3. By Lemma 2 for $\beta \geq 1$, we can write

$$(i)\ \eta_{n,\alpha,\beta}(x,y) = \int_0^y W_{n,\alpha,\beta}(x,t)dt \leq \frac{\beta\lambda x(1+\alpha x)}{n(x-y)^2},\ 0 \leq y < x$$

$$(ii)\ 1 - \eta_{n,\alpha,\beta}(x,z) = \int_z^\infty W_{n,\alpha,\beta}(x,t)dt \leq \frac{\beta\lambda x(1+\alpha x)}{n(z-x)^2},\ x < z < \infty.$$

Remark 4. In view of Lemma 1 and Remark 1, for $\beta \geq 1$, it can be observed that

$$V_{n,\alpha,\beta}((t-x),x) \leq \frac{\alpha\beta x}{n-\alpha}$$

$$V_{n,\alpha,\beta}(| t-x |, x) \leq [\beta T_{n,2,\alpha}(x)]^{1/2} \leq \sqrt{\frac{\beta\lambda x(1+\alpha x)}{n}}.$$

Our main result is stated as follows:

Theorem 2. *Let $f \in DB_\gamma(0,\infty), \gamma > 0$, also suppose $\alpha > 0, \beta \geq 1$, and $x \in (0,\infty)$. Then for $\lambda > 2$ and n sufficiently large, we have*

$$
\begin{aligned}
| V_{n,\alpha,\beta}(f,x) - f(x) | \leq{}& \frac{\beta\lambda(1+\alpha x)}{n}\left(\sum_{v=1}^{[\sqrt{n}]}\bigvee_{x-x/v}^{x+x/v}((f')_x) + \frac{x}{\sqrt{n}}\bigvee_{x-x/\sqrt{n}}^{x+x/\sqrt{n}}((f')_x)\right) \\
&+ \frac{\beta\lambda(1+\alpha x)}{nx}\left(| f(2x) - f(x) - xf'(x^+) | + | f(x) |\right) \\
&+ C\sqrt{\frac{\lambda x(1+\alpha x)}{n}}(2^\gamma O(n^{-\gamma/2}) + | f'(x^+) |) \\
&+ \frac{\beta}{\beta+1}\sqrt{\frac{\beta\lambda x(1+\alpha x)}{n}}\ | f('x^+) - f'(x^-) | \\
&+ \frac{1}{\beta+1}[f'(x^+) + \beta f'(x^-)]\frac{\alpha\beta x}{(n-\alpha)},
\end{aligned}
$$

where $\bigvee_a^b f(x)$ denotes the total variation of f_x on $[a,b]$ and the auxiliary function f_x is as given in Theorem 1.

Proof. Following the methods presented in [7], we can write

$$
\begin{aligned}
&| V_{n,\alpha,\beta}(f,x) - f(x) | \\
&\leq \left| \int_x^\infty \left(\int_x^t (f')_x(u)du\right)W_{n,\alpha,\beta}(x,t)dt - \int_0^x \left(\int_x^t (f')_x(u)du\right)W_{n,\alpha,\beta}(x,t)dt \right| \\
&\quad + \frac{\beta}{\beta+1} | f'(x^+) - f'(x^-) | V_{n,\alpha,\beta}(| t-x |, x) \\
&\quad + \frac{1}{\beta+1}[f'(x^+) + \beta f'(x^-)]V_{n,\alpha,\beta}((t-x),x) \\
&= | A_{n,\alpha,\beta}(f,x) + B_{n,\alpha,\beta}(f,x) | \\
&\quad + \frac{\beta}{\beta+1}|f'(x^+) - f'(x^-)|V_{n,\alpha,\beta}(| t-x |, x) \\
&\quad + \frac{1}{\beta+1}[f'(x^+) + \beta f'(x^-)]V_{n,\alpha,\beta}((t-x),x).
\end{aligned}
$$

Using Remark 4, we have

$$
\begin{aligned}
| V_{n,\alpha,\beta}(f,x) - f(x) | &\leq | A_{n,\alpha,\beta}(f,x) + B_{n,\alpha,\beta}(f,x) | \\
+ \frac{\beta}{\beta+1}|f'(x^+) - f'(x^-)|\sqrt{\frac{\beta\lambda x(1+\alpha x)}{n}} &+ \frac{1}{\beta+1}[f'(x^+) + \beta f'(x^-)]\frac{\alpha\beta x}{(n-\alpha)}. \quad (9)
\end{aligned}
$$

In order to complete the proof of the theorem, it is sufficient to estimate the terms $A_{n,\alpha,\beta}(f,x)$ and $B_{n,\alpha,\beta}(f,x)$ given in (9) above. Using Remark 3 and proceeding along the lines of proof of Theorem 1, we get the desired estimate. Here we omit the details.

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