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HECKE OPERATORS ON COHOMOLOGY

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Abstract. Hecke operators play an important role in the theory of automorphic forms, and automorphic forms are closely linked to various cohomology groups. This paper is mostly a survey of Hecke operators acting on certain types of cohomology groups. The class of cohomology on which Hecke operators are introduced includes the group cohomology of discrete subgroups of a semisimple Lie group, the de Rham cohomology of locally symmetric spaces, and the cohomology of symmetric spaces with coefficients in a system of local groups. We construct canonical isomorphisms among such cohomology groups and discuss the compatibility of the Hecke operators with respect to those canonical isomorphisms.

1. Introduction

This paper is mainly a survey of Hecke operators acting on certain types of cohomology groups. The class of cohomology on which Hecke operators are introduced includes the group cohomology of discrete subgroups of a semisimple Lie group, the de Rham cohomology of locally symmetric spaces, and the cohomology of symmetric spaces with coefficients in a system of local groups. We construct canonical isomorphisms among such cohomology groups and discuss the compatibility of the Hecke operators with respect to those canonical isomorphisms.

Automorphic forms play a major role in number theory, and they are closely related to many other areas of mathematics. Modular forms, or automorphic forms of one variable, are holomorphic functions on the Poincaré upper half plane $\mathcal{H}$ satisfying a certain transformation formula with respect to the linear fractional action of a discrete subgroup Γ of $SL(2,\mathbb{R})$, and they are closely linked to the geometry of the associated Riemann surface $X = \Gamma\backslash\mathcal{H}$. For example, modular forms for Γ can be interpreted as holomorphic sections of a line bundle over X, and the space of such modular forms of a given weight corresponds to a certain cohomology group of X with local coefficients or with some cohomology group of the discrete group Γ (cf. [1], [2], [5]) with coefficients in some Γ-module. Modular forms can be extended to automorphic forms of several variables by using holomorphic functions either on the Cartesian product $\mathcal{H}^n$ of n copies of $\mathcal{H}$ for Hilbert modular forms or on the Siegel upper half space $\mathcal{H}_n$ of degree n for Siegel modular forms. More general automorphic forms can also be considered by using semisimple Lie groups. Indeed, given a semisimple Lie group G of Hermitian type and a discrete subgroup Γ of G, we can consider automorphic forms for Γ defined on the quotient

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$\mathcal{D} = G/K$ of G by a maximal compact subgroup K of G. The space $\mathcal{D}$ has the structure of a Hermitian symmetric domain, and automorphic forms on $\mathcal{D}$ for Γ are holomorphic functions on $\mathcal{D}$ satisfying an appropriate transformation formula with respect to the natural action of Γ on $\mathcal{D}$ (cf. [3]). Such automorphic forms are also linked to families of abelian varieties parametrized by the locally symmetric space $\Gamma\backslash\mathcal{D}$ (cf. [6], [10], [14]). Close connections between automorphic forms for the discrete group $\Gamma \subset G$ and the group cohomology of Γ or the de Rham cohomology of $\mathcal{D}$ with certain coefficients have also been studied in numerous papers over the years (see e.g. [11]).

Hecke operators are certain averaging operators acting on the space of automorphic forms (cf. [1], [12], [15]), and they are an important component of the theory of automorphic forms. For example, they are used to obtain Euler products associated to modular forms which lead to some multiplicative properties of Fourier coefficients of those automorphic forms. In light of the fact that automorphic forms are closely related to the cohomology of the corresponding discrete subgroups of a semisimple Lie group, it would be natural to study the Hecke operators on the cohomology of the discrete groups associated to automorphic forms as was done in a number of papers (see e.g. [6], [8], [7], [17]). Hecke operators on the cohomology of more general groups were also investigated by Rhie and Whaples in [13]. On the other hand, if f is an automorphic form on a Hermitian symmetric domain $\mathcal{D} = G/K$ for a discrete subgroup Γ of G described above, then f can be interpreted as an algebraic correspondence on the quotient space $\Gamma\backslash\mathcal{D}$, which has the structure of a complex manifold, assuming that Γ is torsion-free. Such a correspondence is determined by a pair of holomorphic maps $\lambda, \mu : \Gamma'\backslash\mathcal{D} \to \Gamma\backslash\mathcal{D}$, where Γ' is another discrete subgroup of G. The maps λ and μ can be used to construct a Hecke operator on the de Rham cohomology of $\Gamma\backslash\mathcal{D}$. The idea of Hecke operators on cohomology of complex manifolds of the kind described above was suggested, for example, by Kuga and Sampson in [9] (see also [7]).

The goal of this paper is to discuss relations among different types of cohomology described above and establish the compatibility of the Hecke operators acting on those cohomology groups. The organization of the paper is as follows. In Section 2 we review Hecke algebras associated to subgroups of a given group, whose examples include the algebras of Hecke operators considered in the subsequent sections. In Section 3 we describe the cohomology of groups as well as Hecke operators acting on such cohomology. We also discuss equivariant cohomology and its relation with group cohomology. The de Rham cohomology of a locally symmetric space with coefficients in a vector bundle is discussed in Section 4 by using the language of sheaves, and then Hecke operators are introduced on Rham cohomology groups. In Section 5 we study the cohomology of a locally symmetric space with coefficients in a local system of groups in connection with other types of cohomology. Hecke operators are also considered for this cohomology. Section 6 is concerned with compatibility of Hecke operators. We discuss canonical isomorphisms among de Rham, singular, and group cohomology and show that the Hecke operators acting on those cohomology groups are compatible with one another under those canonical isomorphisms.

2. Hecke algebras

In this section we review some of the basic properties of Hecke algebras. In Section 2.1 we discuss the commensurability relation on the set of subgroups of a given group G, consider double cosets determined by two commensurable subgroups of G, and describe decompositions of such double cosets in terms of left or right cosets of one of those two subgroups. We introduce a binary operation on the set of double cosets in Section 2.2, which is used in Section 2.3 to construct the structure of an algebra, known as a Hecke algebra, on the set of double cosets determined by a single subgroup of the given group. More details and some additional properties of Hecke algebras can be found, for example, in [6], [12] and [15].

2.1. Double cosets.

Let G be a group. Two subgroups Γ and Γ' are said to be *commensurable* (or Γ is said to be commensurable with Γ') if

$$[\Gamma : \Gamma \cap \Gamma'] < \infty, \quad [\Gamma' : \Gamma \cap \Gamma'] < \infty,$$

that is, if $\Gamma \cap \Gamma'$ has finite index in both Γ and Γ'. We shall write $\Gamma \sim \Gamma'$ when Γ is commensurable with Γ'. If H is a subgroup of G and if K is a subset of G containing H, then we shall denote by K/H (resp. $H\backslash K$) the set of left (resp. right) cosets of H in K.

Lemma 2.1. *The commensurability relation $\sim$ is an equivalence relation.*

Proof. The relation $\sim$ is clearly reflexive and symmetric. Let Γ_1, Γ_2 and Γ_3 be subgroups of G with $\Gamma_1 \sim \Gamma_2$ and $\Gamma_2 \sim \Gamma_3$. We consider the map

$$\Gamma_1 \cap \Gamma_2 / \Gamma_1 \cap \Gamma_2 \cap \Gamma_3 \to \Gamma_2 / \Gamma_2 \cap \Gamma_3 \tag{2.1}$$

sending the left coset $\gamma(\Gamma_1 \cap \Gamma_2 \cap \Gamma_3)$ to the left coset $\gamma(\Gamma_2 \cap \Gamma_3)$ for each $\gamma \in \Gamma_1 \cap \Gamma_2$. If $\gamma(\Gamma_2 \cap \Gamma_3) = \gamma'(\Gamma_2 \cap \Gamma_3)$ with $\gamma, \gamma' \in \Gamma_1 \cap \Gamma_2$, then $\gamma^{-1}\gamma' \in \Gamma_1 \cap \Gamma_2 \cap \Gamma_3$; hence we see that $\gamma(\Gamma_1 \cap \Gamma_2 \cap \Gamma_3) = \gamma'(\Gamma_1 \cap \Gamma_2 \cap \Gamma_3)$. Thus the map (2.1) is injective, and therefore we have

$$[\Gamma_1 \cap \Gamma_2 : \Gamma_1 \cap \Gamma_2 \cap \Gamma_3] \leq [\Gamma_2 : \Gamma_2 \cap \Gamma_3] < \infty,$$

which implies that

$$[\Gamma_1 : \Gamma_1 \cap \Gamma_2 \cap \Gamma_3] = [\Gamma_1 : \Gamma_1 \cap \Gamma_2][\Gamma_1 \cap \Gamma_2 : \Gamma_1 \cap \Gamma_2 \cap \Gamma_3] < \infty.$$

Similarly, it can be shown that

$$[\Gamma_3 : \Gamma_1 \cap \Gamma_2 \cap \Gamma_3] < \infty,$$

and hence we obtain

$$[\Gamma_1 : \Gamma_1 \cap \Gamma_3] \leq [\Gamma_1 : \Gamma_1 \cap \Gamma_2 \cap \Gamma_3] < \infty,$$
$$[\Gamma_3 : \Gamma_1 \cap \Gamma_3] \leq [\Gamma_3 : \Gamma_1 \cap \Gamma_2 \cap \Gamma_3] < \infty.$$

Thus the relation is transitive, and therefore the lemma follows. □

Given a subgroup Γ of G, we set

$$\widetilde{\Gamma} = \{\alpha \in G \mid \alpha^{-1}\Gamma\alpha \sim \Gamma\},$$

which will be called the *commensurator* of Γ in G.

Lemma 2.2. *The commensurator $\widetilde{\Gamma}$ is a subgroup of G containing Γ.*

Proof. Given $\alpha, \beta \in \widetilde{\Gamma}$, since $\alpha^{-1}\Gamma\alpha \sim \Gamma$, we see that

$$(\alpha\beta^{-1})^{-1}\Gamma(\alpha\beta^{-1}) = \beta(\alpha^{-1}\Gamma\alpha)\beta^{-1}$$

is commensurable with $\beta\Gamma\beta^{-1}$. However, the commensurability $\beta^{-1}\Gamma\beta \sim \Gamma$ implies that $\Gamma \sim \beta\Gamma\beta^{-1}$; hence we have

$$(\alpha\beta^{-1})^{-1}\Gamma(\alpha\beta^{-1}) \sim \Gamma.$$

Thus $\alpha\beta^{-1} \in \widetilde{\Gamma}$, and therefore $\widetilde{\Gamma}$ is a subgroup of G. Since $\widetilde{\Gamma}$ clearly contains Γ, the proof of the lemma is complete. □

Lemma 2.3. *If $\Gamma \sim \Gamma'$, then $\widetilde{\Gamma}' = \widetilde{\Gamma}$.*

Proof. If $\Gamma \sim \Gamma'$ and $\alpha \in \Gamma'$, then we have

$$\alpha^{-1}\Gamma\alpha \sim \alpha^{-1}\Gamma'\alpha = \Gamma' \sim \Gamma;$$

hence $\alpha \in \widetilde{\Gamma}$, which shows that $\Gamma' \subset \widetilde{\Gamma}$. On the other hand, if $\beta \in \widetilde{\Gamma}'$, then we have

$$\beta^{-1}\Gamma\beta \sim \beta^{-1}\Gamma'\beta \sim \Gamma' \sim \Gamma;$$

hence $\beta \in \widetilde{\Gamma}$. Thus we have $\widetilde{\Gamma}' \subset \widetilde{\Gamma}$. Similarly, it can be shown that $\widetilde{\Gamma} \subset \widetilde{\Gamma}'$, and therefore we obtain $\widetilde{\Gamma}' = \widetilde{\Gamma}$. □

Proposition 2.4. *Let $\Gamma \sim \Gamma'$, and let $\alpha \in \widetilde{\Gamma}$. Then the double coset $\Gamma\alpha\Gamma'$ can be decomposed into disjoint unions of the form*

$$\Gamma\alpha\Gamma' = \coprod_{i=1}^{r} \Gamma\alpha\gamma_i = \coprod_{j=1}^{s} \delta_j\alpha\Gamma' \tag{2.2}$$

for some positive integers r and s, where $\{\gamma_i\}_{i=1}^r$ and $\{\delta_j\}_{j=1}^s$ are complete sets of coset representatives of $(\Gamma' \cap \alpha^{-1}\Gamma\alpha)\backslash\Gamma'$ and $\Gamma/(\Gamma \cap \alpha^{-1}\Gamma'\alpha)$, respectively.

Proof. We note first that a right coset of Γ contained in $\Gamma\alpha\Gamma'$ can be written in the form $\Gamma\alpha\gamma$ for some $\gamma \in \Gamma'$. If $\Gamma\alpha\gamma'$ with $\gamma' \in \Gamma'$ is another subset of $\Gamma\alpha\Gamma'$, we see that $\Gamma\alpha\gamma = \Gamma\alpha\gamma'$ if and only if $\gamma'\gamma^{-1} \in \Gamma' \cap \alpha^{-1}\Gamma\alpha$, which is equivalent to the condition that

$$(\Gamma' \cap \alpha^{-1}\Gamma\alpha)\gamma = (\Gamma' \cap \alpha^{-1}\Gamma\alpha)\gamma'.$$

Since $\alpha^{-1}\Gamma\alpha \sim \Gamma \sim \Gamma'$, the index $[\Gamma' : \Gamma' \cap \alpha^{-1}\Gamma\alpha]$ is finite. Thus, if $\{\gamma_i\}_{i=1}^r$ is a set of representatives of $(\Gamma' \cap \alpha^{-1}\Gamma\alpha)\backslash\Gamma'$, each γ_i determines a unique coset $\Gamma\alpha\gamma_i$ contained in $\Gamma\alpha\Gamma'$; hence we have

$$\Gamma\alpha\Gamma' = \coprod_{i=1}^{r} \Gamma\alpha\gamma_i.$$

Similarly, it can be shown that

$$\Gamma\alpha\Gamma' = \coprod_{j=1}^{s} \delta_j \alpha\Gamma',$$

where $s = [\Gamma : \Gamma \cap \alpha^{-1}\Gamma'\alpha]$. □

2.2. **Operations on double cosets.** Let G be the group considered in Section 2.1, and fix a subsemigroup Δ of G. We denote by $\mathcal{C}(\Delta)$ the collection of subgroups Γ of G that are mutually commensurable and satisfy

$$\Gamma \subset \Delta \subset \widetilde{\Gamma}.$$

Given $\Gamma, \Gamma' \in \mathcal{C}(\Delta)$ and a commutative ring R be with identity, we denote by $\mathcal{H}_R(\Gamma, \Gamma'; \Delta)$ the free R-module generated by the double cosets $\Gamma\alpha\Gamma'$ with $\alpha \in \Delta$. Thus an element of $\mathcal{H}_R(\Gamma, \Gamma'; \Delta)$ can be written in the form

$$\sum_{\alpha\in\Delta} c_\alpha \Gamma\alpha\Gamma',$$

where the coefficients $c_\alpha \in R$ are zero except for a finite number of α. We denote by $\deg(\Gamma\alpha\Gamma')$ the number of right cosets $\Gamma\gamma$ contained in $\Gamma\alpha\Gamma'$. Thus, if $\Gamma\alpha\Gamma'$ is as in (2.2), then $\deg(\Gamma\alpha\Gamma') = r$. If η is an element of $\mathcal{H}_R(\Gamma, \Gamma'; \Delta)$ given by $\eta = \sum_{\alpha\in\Delta} c_\alpha \Gamma\alpha\Gamma'$, then we set

$$\deg \eta = \sum_{\alpha\in\Delta} c_\alpha \deg(\Gamma\alpha\Gamma') \tag{2.3}$$

and refer to it as the *degree* of η.

We now consider an R-module M and assume that the subsemigroup $\Delta \subset G$ acts on M on the right by

$$(m, \delta) \mapsto m \cdot \delta \in M$$

for $(m, \delta) \in M \times \Delta$. Thus we have

$$m \cdot 1 = m, \quad m \cdot (\delta\delta') = (m \cdot \delta) \cdot \delta'$$

for all $m \in M$ and $\delta, \delta' \in \Delta$. Given $\Gamma \in \mathcal{C}(\Delta)$, let M^Γ denote the submodule of M consisting of the Γ-invariant elements of M, that is,

$$M^\Gamma = \{m \in M \mid m \cdot \gamma = m \text{ for all } \gamma \in \Gamma\}.$$

If the double coset $\Gamma\alpha\Gamma'$ with $\alpha \in \Delta$ and $\Gamma, \Gamma' \in \mathcal{C}(\Delta)$ has a decomposition of the form

$$\Gamma\alpha\Gamma' = \coprod_{i=1}^{d} \Gamma\alpha_i, \tag{2.4}$$

then we define its operation on M^Γ by

$$m \mid \Gamma\alpha\Gamma' = \sum_{i=1}^{d} m \cdot \alpha_i \tag{2.5}$$

for all $m \in M^\Gamma$.

Lemma 2.5. *The operation of* $\Gamma\alpha\Gamma'$ *on* M^Γ *in* (2.15) *is independent of the choice of the representatives* α_i *of the right cosets of* Γ *in* (2.4) *and*

$$m \mid \Gamma\alpha\Gamma' \in M^{\Gamma'}$$

for all $m \in M^\Gamma$.

Proof. If $\Gamma\alpha_i, \Gamma\alpha_i'$ are subsets of $\Gamma\alpha\Gamma'$ with $\Gamma\alpha_i = \Gamma\alpha_i'$, then $\alpha_i' = \gamma\alpha_i$ for some $\gamma \in \Gamma$. Thus we see that $m \cdot \alpha_i' = (m \cdot \gamma) \cdot \alpha_i = m \cdot \alpha_i$ for all $m \in M^\Gamma$; hence $m \mid \Gamma\alpha\Gamma'$ is independent of the choice of the representatives α_i. On the other hand, if $\Gamma\alpha\Gamma'$ has a decomposition as in (2.4), then we see that

$$\Gamma\alpha\Gamma' = \coprod_{i=1}^{d} \Gamma\alpha_i\gamma'$$

for all $\gamma' \in \Gamma'$. Thus we have

$$(m \mid \Gamma\alpha\Gamma') \cdot \gamma' = \sum_{i=1}^{d} m \cdot (\alpha_i\gamma') = \sum_{i=1}^{d} m \cdot \alpha_i = m \mid \Gamma\alpha\Gamma';$$

hence it follows that $m \mid \Gamma\alpha\Gamma' \in M^{\Gamma'}$. □

We see easily that the map $m \mapsto (m \mid \Gamma\alpha\Gamma')$ given by (2.15) is in fact a homomorphism of R-modules. We now extend this by defining an R-module homomorphism associated to each element of $\mathcal{H}_R(\Gamma, \Gamma'; \Delta)$ by

$$m \mid \eta = \sum_{\alpha} c_\alpha (m \mid \Gamma\alpha\Gamma')$$

for $m \in M^\Gamma$ and $\eta = \sum_\alpha c_\alpha \Gamma\alpha\Gamma' \in \mathcal{H}_R(\Gamma, \Gamma'; \Delta)$.

Given elements $\Gamma_1, \Gamma_2, \Gamma_3 \in \mathcal{C}(\Delta)$ and double cosets of the form

$$\Gamma_1\alpha\Gamma_2 = \coprod_{i=1}^{r} \Gamma_1\alpha_i, \quad \Gamma_2\beta\Gamma_3 = \coprod_{j=1}^{s} \Gamma_2\beta_j \tag{2.6}$$

with $\alpha, \beta \in \Delta$, we set

$$(\Gamma_1\alpha\Gamma_2) \cdot (\Gamma_2\beta\Gamma_3) = \sum_{\gamma} c_\gamma \Gamma_1\gamma\Gamma_3, \tag{2.7}$$

where the summation is over the set of representatives $\gamma \in \Delta$ of the double cosets $\Gamma_1\gamma\Gamma_3$ contained in Δ and

$$c_\gamma = \#\{(i,j) \mid \Gamma_1\alpha_i\beta_j = \Gamma_1\gamma\} \tag{2.8}$$

is the number of pairs (i,j) with $1 \le i \le r$ and $1 \le j \le s$ such that $\Gamma_1\alpha_i\beta_j = \Gamma_1\gamma$. Since $c_\gamma = 0$ except for a finitely many double cosets $\Gamma_1\gamma\Gamma_3$, the sum on the right hand side of (2.7) is a finite sum.

Let $R[\Gamma_1\backslash\Delta]$ denote the free R-module generated by the right cosets $\Gamma_1\alpha$ with $\alpha \in \Delta$. Then Δ acts on $R[\Gamma_1\backslash\Delta]$ by right multiplication. On the other hand, there is a natural injective map $\mathcal{H}_R(\Gamma_1, \Gamma_2; \Delta) \to R[\Gamma_1\backslash\Delta]$ sending $\Gamma_1\alpha\Gamma_2 = \coprod_i \Gamma_1\alpha_i$ to

$\sum_i \Gamma_1\alpha_i$. By using this injection we may regard $\mathcal{H}_R(\Gamma_1, \Gamma_2; \Delta)$ as an R-submodule of $R[\Gamma_1\backslash\Delta]$, and under this identification we see easily that

$$\mathcal{H}_R(\Gamma_1, \Gamma_2; \Delta) = R[\Gamma_1\backslash\Delta]^{\Gamma_2}. \tag{2.9}$$

If the double cosets $\Gamma_1\alpha\Gamma_2$ and $\Gamma_2\beta\Gamma_3$ are as in (2.6), using (2.15) and (2.8), we have

$$(\Gamma_1\alpha\Gamma_2) \mid (\Gamma_2\beta\Gamma_3) = \sum_{i=1}^{r} \Gamma_1\alpha_i \mid (\Gamma_2\beta\Gamma_3) = \sum_{i=1}^{r}\sum_{j=1}^{s} \Gamma_1\alpha_i\beta_j = \sum_{\gamma} c_\gamma \Gamma_1\gamma.$$

Using Lemma 2.5 and the identification (2.9) with Γ, Γ' replaced by Γ_2, Γ_3, we see that

$$\sum_{\gamma} c_\gamma \Gamma_1\gamma \in R[\Gamma_1\backslash\Delta]^{\Gamma_3}.$$

Thus by using (2.9) again, we obtain

$$\sum_{\gamma} c_\gamma \Gamma_1\gamma = \sum_{\gamma} c_\gamma \Gamma_1\gamma\Gamma_3;$$

hence it follows that

$$(\Gamma_1\alpha\Gamma_2) \cdot (\Gamma_2\beta\Gamma_3) = (\Gamma_1\alpha\Gamma_2) \mid (\Gamma_2\beta\Gamma_3). \tag{2.10}$$

From this and Lemma 2.5 we see that the operation in (2.7) is independent of the choice of the representatives α_i, β_j and γ.

Lemma 2.6. *Let $\Gamma_1\alpha\Gamma_2$ and $\Gamma_2\beta\Gamma_3$ be as in (2.6), and let c_γ with $\Gamma_1\gamma\Gamma_3 \subset \Delta$ be as in (2.8). Then we have*

$$c_\gamma \deg(\Gamma_1\gamma\Gamma_3) = \#\{(i,j) \mid \Gamma_1\alpha_i\beta_j\Gamma_3 = \Gamma_1\gamma\Gamma_3\}$$

for each $\gamma \in \Delta$.

Proof. We assume that $\Gamma_1\gamma\Gamma_3 \subset \Delta$ has a decomposition of the form

$$\Gamma_1\gamma\Gamma_3 = \coprod_{k=1}^{t} \Gamma_1\gamma_k.$$

Then the relation $\Gamma_1\alpha_i\beta_j\Gamma_3 = \Gamma_1\gamma\Gamma_3$ holds if and only if $\Gamma_1\alpha_i\beta_j = \Gamma_1\gamma_k$ for exactly one $k \in \{1, \dots, t\}$. Thus, if c_γ is as in (2.8), we see that

$$\#\{(i,j) \mid \Gamma_1\alpha_i\beta_j\Gamma_3 = \Gamma_1\gamma\Gamma_3\} = \sum_{k=1}^{t} \#\{(i,j) \mid \Gamma_1\alpha_i\beta_j = \Gamma_1\gamma\} = c_\gamma t;$$

hence the lemma follows from this and the fact that $\deg(\Gamma_1\gamma\Gamma_2) = t$. ∎

Lemma 2.7. *If $\eta_1 \in \mathcal{H}_R(\Gamma_1, \Gamma_2; \Delta)$ and $\eta_2 \in \mathcal{H}_R(\Gamma_2, \Gamma_3; \Delta)$, then we have*

$$\deg(\eta_1 \cdot \eta_2) = \deg(\eta_1)\deg(\eta_2).$$

Proof. Let $\Gamma_1\alpha\Gamma_2 \in \mathcal{H}_R(\Gamma_1, \Gamma_2; \Delta)$ and $\Gamma_2\beta\Gamma_3 \in \mathcal{H}_R(\Gamma_2, \Gamma_3; \Delta)$ be as in (2.7). Then, using (2.3) and Lemma 2.6, we have

$$\deg[(\Gamma_1\alpha\Gamma_2) \cdot (\Gamma_2\beta\Gamma_3)] = \sum_\gamma c_\gamma \deg(\Gamma_1\gamma\Gamma_3).$$

However, by (2.8) the right hand side of this relation is equal to the number of pairs (i, j) with $1 \le i \le r$ and $1 \le j \le s$ and therefore is equal to $rs = \deg(\Gamma_1\alpha\Gamma_2) \cdot \deg(\Gamma_2\alpha\Gamma_3)$. Thus the lemma follows by extending this result linearly. □

2.3. **Hecke algebras.** Given $\Gamma_1, \Gamma_2, \Gamma_3 \in \mathcal{C}(\Delta)$, the operation in (2.7) induces a bilinear map

$$\mathcal{H}_R(\Gamma_1, \Gamma_2; \Delta) \times \mathcal{H}_R(\Gamma_2, \Gamma_3; \Delta) \to \mathcal{H}_R(\Gamma_1, \Gamma_3; \Delta)$$

defined by

$$\left(\sum_\alpha a_\alpha \Gamma_1\alpha\Gamma_2\right) \cdot \left(\sum_\beta b_\beta \Gamma_2\beta\Gamma_3\right) = \sum_{\alpha,\beta} a_\alpha b_\beta (\Gamma_1\alpha\Gamma_2) \cdot (\Gamma_2\beta\Gamma_3). \tag{2.11}$$

Using (2.10), we see that the operation of $\mathcal{H}_R(\Gamma_2, \Gamma_3; \Delta)$ on $\mathcal{H}_R(\Gamma_1, \Gamma_2; \Delta) = R[\Gamma_1\backslash\Delta]^{\Gamma_2}$ coincides with the multiplication operation in (2.11), that is,

$$\eta_1 \cdot \eta_2 = \eta_1 \mid \eta_2 \tag{2.12}$$

for all $\eta_1 \in \mathcal{H}_R(\Gamma_1, \Gamma_2; \Delta)$ and $\eta_2 \in \mathcal{H}_R(\Gamma_2, \Gamma_3; \Delta)$.

If M is an R-module on which Δ acts on the right, then it follows easily from the definition that

$$(m \mid \eta_1) \mid \eta_2 = m \mid (\eta_1 \cdot \eta_2)$$

for all $m \in M^{\Gamma_1}$, $\eta_1 \in \mathcal{H}_R(\Gamma_1, \Gamma_2; \Delta)$ and $\eta_2 \in \mathcal{H}_R(\Gamma_2, \Gamma_3; \Delta)$. From this and (2.12) we obtain

$$(\eta_1 \cdot \eta_2) \cdot \eta_3 = \eta_1 \cdot (\eta_2 \cdot \eta_3) \tag{2.13}$$

for all $\eta_1 \in \mathcal{H}_R(\Gamma_1, \Gamma_2; \Delta)$, $\eta_2 \in \mathcal{H}_R(\Gamma_2, \Gamma_3; \Delta)$ and $\eta_3 \in \mathcal{H}_R(\Gamma_3, \Gamma_4; \Delta)$.

Given $\Gamma \in \mathcal{C}(\Delta)$, we set

$$\mathcal{H}_R(\Gamma; \Delta) = \mathcal{H}_R(\Gamma, \Gamma; \Delta).$$

Then by (2.13) the multiplication operation on $\mathcal{H}_R(\Gamma; \Delta)$ is associative and $\mathcal{H}_R(\Gamma; \Delta)$ is an algebra over R with identity Γ. When $R = \mathbb{Z}$, we shall simply write

$$\mathcal{H}(\Gamma; \Delta) = \mathcal{H}_{\mathbb{Z}}(\Gamma; \Delta) = \mathcal{H}_{\mathbb{Z}}(\Gamma, \Gamma; \Delta).$$

Definition 2.8. Given $\Gamma \in \mathcal{C}(\Delta)$, the algebra $\mathcal{H}_R(\Gamma; \Delta)$ is called the *Hecke algebra over* R of Γ with respect to Δ. If $R = \mathbb{Z}$, then $\mathcal{H}(\Gamma; \Delta) = \mathcal{H}_{\mathbb{Z}}(\Gamma; \Delta)$ is simply called the *Hecke algebra* of Γ with respect to Δ.

Let Δ and Δ' be two subsemigroups of G with $\Delta \subset \Delta'$. Then certainly $\mathcal{H}_R(\Gamma; \Delta)$ is a subset of $\mathcal{H}_R(\Gamma; \Delta')$. If $\Gamma\alpha\Gamma, \Gamma\beta\Gamma \in \mathcal{H}_R(\Gamma; \Delta)$ with $\alpha, \beta \in \Delta$ are regarded as elements of $\mathcal{H}_R(\Gamma; \Delta')$, their product can be written in the form

$$(\Gamma\alpha\Gamma) \cdot (\Gamma\beta\Gamma) = \sum_\gamma c'_\gamma \Gamma\gamma\Gamma, \tag{2.14}$$

where the summation is over the set of representatives γ of the double cosets $\Gamma\gamma\Gamma$ contained in Δ'. However, we have $\Gamma\gamma\Gamma \subset \Gamma\alpha\Gamma\beta\Gamma \subset \Delta$; hence the product in (2.14) coincides with the product of $\Gamma\alpha\Gamma$ and $\Gamma\beta\Gamma$ in $\mathcal{H}_R(\Gamma;\Delta)$. Thus we see that $\mathcal{H}_R(\Gamma;\Delta)$ is a subalgebra of $\mathcal{H}_R(\Gamma;\Delta')$.

Proposition 2.9. *Let $\alpha \in \widetilde{\Gamma}$, and assume that $|\Gamma\backslash\Gamma\alpha\Gamma| = |\Gamma\alpha\Gamma/\Gamma|$. Then the quotients $\Gamma\backslash\Gamma\alpha\Gamma$ and $\Gamma\alpha\Gamma/\Gamma$ have a common set of coset representatives.*

Proof. We assume that $\Gamma\alpha\Gamma$ can be decomposed as

$$\Gamma\alpha\Gamma = \coprod_{i=1}^{d} \Gamma\alpha_i = \coprod_{i=1}^{d} \beta_i\Gamma.$$

Then it can be shown that $\Gamma\alpha_i \cap \beta_j\Gamma$ is nonempty for all i and j. Indeed, if $\Gamma\alpha_i$ and $\beta_i\Gamma$ are disjoint for some i and j, then $\Gamma\alpha_i \subset \bigcup_{k\neq j} \beta_k\Gamma$, and therefore we have

$$\Gamma\alpha\Gamma = \Gamma\alpha_i\Gamma = \bigcup_{k\neq j} \beta_k\Gamma,$$

which is a contradiction. Thus, in particular, we have $\Gamma\alpha_i \cap \beta_i\Gamma \neq \emptyset$ for each i. If $\delta_i \in \Gamma\alpha_i \cap \beta_i\Gamma$ for each i, then we see that $\Gamma\alpha_i = \Gamma\delta_i$ and $\delta_i\Gamma = \beta_i\Gamma$. Hence we have

$$\Gamma\alpha\Gamma = \coprod_{i=1}^{d} \Gamma\delta_i = \coprod_{i=1}^{d} \delta_i\Gamma,$$

and $\{\delta_i\}_{i=1}^d$ is a common set of coset representatives. □

We now discuss the commutativity of the Hecke algebra $\mathcal{H}_R(\Gamma;\Delta)$. Note that an involution on Δ is a map $\iota : \Delta \to \Delta$ satisfying

$$(\alpha\beta)^\iota = \beta^\iota\alpha^\iota, \quad (\alpha^\iota)^\iota$$

for all $\alpha, \beta \in \Delta$.

Theorem 2.10. *Let $\iota : \Delta \to \Delta$ be an involution on Δ, and assume that an element $\Gamma \in \mathcal{C}(\Delta)$ satisfies*

$$\Gamma^\iota = \Gamma, \quad \Gamma\alpha^\iota\Gamma = \Gamma\alpha\Gamma \tag{2.15}$$

for all $\alpha \in \Delta$. Then the associated Hecke algebra $\mathcal{H}_R(\Gamma;\Delta)$ is commutative.

Proof. Given $\alpha \in \Delta$ with $\Gamma\alpha\Gamma = \coprod_{i=1}^d \Gamma\alpha_i$, using (2.15), we have

$$\Gamma\alpha\Gamma = \Gamma\alpha^\iota\Gamma = (\Gamma\alpha\Gamma)^\iota = \coprod_{i=1}^{d} \alpha_i^\iota\Gamma.$$

Hence by Lemma 2.9 the sets $\Gamma\backslash\Gamma\alpha\Gamma$ and $\Gamma\alpha\Gamma/\Gamma$ have a common set of coset representatives. Thus we may write

$$\Gamma\alpha\Gamma = \coprod_{i=1}^{d} \Gamma\alpha_i = \coprod_{i=1}^{d} \alpha_i\Gamma$$

for some $\alpha_1, \dots, \alpha_d \in \Delta$. Similarly, if β is another element of Δ, we have

$$\Gamma\beta\Gamma = \coprod_{i=1}^{s} \Gamma\beta_j = \coprod_{j=1}^{s} \beta_j\Gamma$$

for some positive integer s and $\beta_j \in \Delta$ for $1 \leq j \leq s$. We now assume that

$$(\Gamma\alpha\Gamma) \cdot (\Gamma\beta\Gamma) = \sum_{\gamma} c_\gamma(\Gamma\gamma\Gamma), \quad (\Gamma\beta\Gamma) \cdot (\Gamma\alpha\Gamma) = \sum_{\gamma} c'_\gamma(\Gamma\gamma\Gamma),$$

where c_γ and $c'_\gamma \in \Delta$ are as in (2.8). Then we have

$$\begin{aligned} c_\gamma &= \#\{(i,j) \mid \Gamma\alpha_i\beta_j = \Gamma\gamma\} \\ &= \#\{(i,j) \mid \Gamma\alpha_i\beta_j = \Gamma\gamma\Gamma\}/|\Gamma\backslash\Gamma\gamma\Gamma| \\ &= \#\{(i,j) \mid \Gamma\beta_j^\iota\alpha_i^\iota = \Gamma\gamma^\iota\Gamma\}/|\Gamma\backslash\Gamma\gamma^\iota\Gamma| \\ &= \#\{(i,j) \mid \Gamma\beta_j^\iota\alpha_i^\iota = \Gamma\gamma^\iota\} = c'_\gamma, \end{aligned}$$

where we used the fact that

$$(\Gamma\beta^\iota\Gamma) \cdot (\Gamma\alpha^\iota\Gamma) = (\Gamma\beta\Gamma) \cdot (\Gamma\alpha\Gamma) = \sum_{\gamma} c'_\gamma(\Gamma\gamma\Gamma) = \sum_{\gamma} c'_\gamma(\Gamma\gamma^\iota\Gamma).$$

Hence it follows that $\mathcal{H}_R(\Gamma;\Delta)$ is a commutative algebra. □

Example 2.11. Let $G = GL(n,\mathbb{Q})$ for some positive integer n, and consider the subgroup $\Gamma = SL(n,\mathbb{Z})$ and the subsemigroup

$$\Delta = \{\alpha \in M(n,\mathbb{Z}) \mid \det\alpha > 0\}$$

of G. Then we see that the transposition $\alpha \mapsto {}^t\alpha$ is an involution satisfying

$${}^t\Gamma = \Gamma, \quad \Gamma \subset \Delta \subset \widetilde{\Gamma}.$$

Given $\alpha \in \Delta$, by the elementary divisor theorem the corresponding double coset $\Gamma\alpha\Gamma$ can be written as

$$\Gamma\alpha\Gamma = \Gamma\alpha_d\Gamma$$

for some diagonal matrix $\alpha_d = \mathrm{diag}(d_1, \dots, d_n)$, where the diagonal entries $d_1, \dots, d_n$ are positive integers satisfying $d_i \mid d_{i+1}$ for each i. Hence we see that

$$\Gamma\,{}^t\alpha\Gamma = \Gamma\,{}^t\alpha_d\Gamma = \Gamma\alpha_d\Gamma = \Gamma\alpha\Gamma.$$

Thus by Theorem 2.10 the Hecke algebra $\mathcal{H}(\Gamma;\Delta) = \mathcal{H}_{\mathbb{Z}}(\Gamma;\Delta)$ is commutative.

3. Group cohomology

In this section we review group cohomology and its relation with equivariant cohomology as well as Hecke operators acting on group cohomology. The description of the cohomology of a group G with coefficients in a G-module by using both homogeneous and nonhomogeneous cochains is given in Section 3.1. Given a complex K on which a group Γ acts on the left and a left Γ-module A, in Section 3.2 we construct the associated equivariant cohomology of K with coefficients in A following Eilenberg [4]. We also obtain an isomorphism between this equivariant cohomology and the cohomology of Γ with the same coefficients. We then discuss

Hecke operators acting on group cohomology in Section 3.3 introduced by Rhie and Whaples [13].

3.1. **Cohomology of groups.** Let G be a group, and let M be a left G-module. Thus M is an abelian group on which G acts on the left. Then the cohomology of G with coefficients in M can be described by using either homogeneous or nonhomogeneous cochains.

Given a nonnegative integer q, let $C^q(G, M)$ denote the group consisting of the M-valued functions $f : G^q \to M$ on the q-fold Cartesian product $G^q = G \times \cdots \times G$ of G, called *nonhomogeneous q-cochains.* We then consider the map $\partial : C^q(G, M) \to C^{q+1}(G, M)$ defined by

$$\begin{aligned}(\partial f)(\sigma_1, \dots, \sigma_{q+1}) = \sigma_1 f(\sigma_2, \dots, \sigma_{q+1}) \\ + \sum_{i=1}^{q} (-1)^i f(\sigma_1, \dots, \sigma_{i-1}, \sigma_i\sigma_{i+1}, \dots, \sigma_{q+1}) \\ + (-1)^{q+1} f(\sigma_1, \dots, \sigma_q)\end{aligned} \tag{3.1}$$

for all $f \in C^q(G, M)$ and $(\sigma_1, \dots, \sigma_{q+1}) \in G^{q+1}$. Then ∂ is the coboundary map for nonhomogeneous q-cochains satisfying $\partial^2 = 0$. The associated q-th cohomology group of G with coefficients in M is given by

$$H^q(G, M) = Z^q(G, M)/B^q(G, M),$$

where $Z^q(G, M)$ is the kernel of $\partial : C^q(G, M) \to C^{q+1}(G, M)$ and $B^q(G, M)$ is the image $\partial : C^{q-1}(G, M) \to C^q(G, M)$.

For each $q \geq 0$ we also consider the group $\mathfrak{C}^q(G, M)$ of *homogeneous q-cochains* consisting of the maps $\phi : G^{q+1} \to M$ satisfying

$$\phi(\sigma\sigma_0, \dots, \sigma\sigma_q) = \sigma\phi(\sigma_0, \dots, \sigma_q)$$

for all $\sigma, \sigma_0, \dots, \sigma_q \in G$. We then define the map $\delta : \mathfrak{C}^q(G, M) \to \mathfrak{C}^{q+1}(G, M)$ by

$$(\delta\phi)(\sigma_0, \dots, \sigma_{q+1}) = \sum_{i=0}^{q+1} (-1)^i \phi(\sigma_0, \dots, \sigma_{i-1}, \sigma_{i+1}, \dots, \sigma_{q+1}) \tag{3.2}$$

for all $\phi \in \mathfrak{C}^q(G, M)$ and $(\sigma_0, \dots, \sigma_{q+1}) \in G^{n+2}$, which is the coboundary map for nonhomogeneous q-cochains satisfying $\delta^2 = 0$. Then the corresponding q-th cohomology group of G in M is given by

$$\mathfrak{H}^q(G, M) = \mathfrak{Z}^q(G, M)/\mathfrak{B}^q(G, M),$$

where $\mathfrak{Z}^q(G, M)$ is the kernel of $\partial : \mathfrak{C}^q(G, M) \to \mathfrak{C}^{q+1}(G, M)$ and $\mathfrak{B}^q(G, M)$ is the image $\partial : \mathfrak{C}^{q-1}(G, M) \to \mathfrak{C}^q(G, M)$.

We can establish a correspondence between homogeneous and nonhomogeneous cochains as follows. Given $f \in C^q(G, M)$ and $\phi \in \mathfrak{C}^q(G, M)$, we consider the elements $f_H \in \mathfrak{C}^q(G, M)$ and $\phi_N \in C^q(G, M)$ given by

$$f_H(\sigma_0, \dots, \sigma_q) = \sigma_0 f(\sigma_0^{-1}\sigma_1, \sigma_1^{-1}\sigma_2, \dots, \sigma_{q-1}^{-1}\sigma_q) \tag{3.3}$$

$$\phi_N(\sigma_1, \dots, \sigma_q) = \phi(1, \sigma_1, \sigma_1\sigma_2, \dots, \sigma_1\sigma_2\cdots\sigma_q) \tag{3.4}$$

for all $\sigma_0, \sigma_1, \dots, \sigma_q \in G$. Then we see that

$$\begin{aligned}(f_H)_N(\sigma_1, \dots, \sigma_q) &= f_H(1, \sigma_1, \sigma_1\sigma_2, \dots, \sigma_1\sigma_2 \cdots \sigma_q) = f(\sigma_1, \dots, \sigma_q),\\ (\phi_N)_H(\sigma_0, \dots, \sigma_q) &= \sigma_0 \phi_N(\sigma_0^{-1}\sigma_1, \sigma_1^{-1}\sigma_2, \dots, \sigma_{q-1}^{-1}\sigma_q)\\ &= \sigma_0 \phi(1, \sigma_0^{-1}\sigma_1, \sigma_0^{-1}\sigma_2, \dots, \sigma_0^{-1}\sigma_q) = \phi(\sigma_0, \dots, \sigma_q)\end{aligned}$$

for all $f \in C^q(G, M)$ and $\phi \in \mathfrak{C}^q(G, M)$. Thus, by extending linearly we obtain the linear maps

$$(\cdot)_H : C^q(G, M) \to \mathfrak{C}^q(G, M), \quad (\cdot)_N : \mathfrak{C}^q(G, M) \to C^q(G, M)$$

such that $(\cdot)_H \circ (\cdot)_N$ and $(\cdot)_N \circ (\cdot)_H$ are identity maps on $\mathfrak{C}^q(G, M)$ and $C^q(G, M)$, respectively. The next lemma shows that this correspondence between homogeneous and nonhomogeneous cochains is compatible with the coboundary maps.

Lemma 3.1. *Given a nonnegative integer q, we have*

$$(\partial f)_H = \delta f_H, \quad (\delta\phi)_N = \partial\phi_N$$

for all $f \in C^q(G, M)$ and $\phi \in \mathfrak{C}^q(G, M)$.

Proof. Given elements $\sigma_0, \sigma_1, \dots, \sigma_{q+1} \in G$ and $f \in C^q(G, M)$, using (3.1), (3.2) and (3.3), we have

$$\begin{aligned}(\partial f)_H(\sigma_0, \sigma_1, \dots, \sigma_{q+1}) &= \sigma_0(\partial f)(\sigma_0^{-1}\sigma_1, \sigma_1^{-1}\sigma_2, \dots, \sigma_q^{-1}\sigma_{q+1})\\ &= \sigma_0\sigma_0^{-1}\sigma_1 f(\sigma_1^{-1}\sigma_2, \dots, \sigma_q^{-1}\sigma_{q+1})\\ &\qquad + \sum_{i=1}^{q}(-1)^i \sigma_0 f(\sigma_0^{-1}\sigma_1, \dots, \sigma_{i-2}^{-1}\sigma_{i-1}, \sigma_{i-1}^{-1}\sigma_{i+1}, \dots, \sigma_q^{-1}\sigma_{q+1})\\ &\qquad + (-1)^{q+1}\sigma_0 f(\sigma_0^{-1}\sigma_1, \dots, \sigma_{q-1}^{-1}\sigma_q)\\ &= f_H(\sigma_1, \dots, \sigma_{q+1}) + \sum_{i=1}^{q}(-1)^i f_H(\sigma_0, \dots, \sigma_{i-1}, \sigma_{i+1}, \dots, \sigma_{q+1})\\ &\qquad + (-1)^{q+1} f_H(\sigma_0, \dots, \sigma_q)\\ &= (\delta f_H)(\sigma_0, \sigma_1, \dots, \sigma_{q+1}).\end{aligned}$$

On the other hand, if $\phi \in \mathfrak{C}^{q+1}(G, M)$, by using (3.1), (3.2) and (3.4) we see that

$$\begin{aligned}(\delta\phi)_N(\sigma_1, \ldots, \sigma_{q+1}) &= (\delta\phi)(1, \sigma_1, \sigma_1\sigma_2, \ldots, \sigma_1\sigma_2\cdots\sigma_{q+1})\\ &= \phi(\sigma_1, \sigma_1\sigma_2, \ldots, \sigma_1\sigma_2\cdots\sigma_{q+1})\\ &\quad + \sum_{i=1}^{q}(-1)^i\phi(1, \sigma_1, \ldots, \sigma_1\cdots\sigma_{i-1}, \sigma_1\cdots\sigma_{i+1}, \ldots, \sigma_1\cdots\sigma_{q+1})\\ &\quad + (-1)^{q+1}\phi(1, \sigma_1, \ldots, \sigma_1\cdots\sigma_{q+1})\\ &= \sigma_1\phi_N(\sigma_2, \ldots, \sigma_{q+1}) + \sum_{i=1}^{q}(-1)^i\phi_N(\sigma_1, \ldots, \sigma_{i-1}, \sigma_i\sigma_{i+1}, \ldots, \sigma_{q+1})\\ &\quad + (-1)^{q+1}\phi_N(\sigma_1, \ldots, \sigma_{q+1})\\ &= (\partial\phi_N)(\sigma_1, \ldots, \sigma_{q+1});\end{aligned}$$

hence the lemma follows. □

From Lemma 3.1 we see that the diagram

$$\begin{array}{ccccc} C^q(G,M) & \xrightarrow{(\cdot)_H} & \mathfrak{C}^q(G,M) & \xrightarrow{(\cdot)_N} & C^q(G,M) \\ \partial\downarrow & & \delta\downarrow & & \downarrow\partial \\ C^{q+1}(G,M) & \xrightarrow{(\cdot)_H} & \mathfrak{C}^{q+1}(G,M) & \xrightarrow{(\cdot)_N} & C^{q+1}(G,M) \end{array}$$

is commutative, which implies that there is a canonical isomorphism

$$H^q(G, M) \cong \mathfrak{H}^q(G, M)$$

for each $q \geq 0$.

3.2. **Equivariant cohomology.** Let K be a complex, which can be described as follows. The elements of the complex K are called *cells*, and there is a nonnegative integer associated to each cell called the *dimension* of the cell. A cell $\sigma_q \in K$ of dimension $q \geq 0$ is referred to as a q-cell, and the *incidence number* $[\sigma_{q+1} : \sigma_q]$ associated to the a q-cell σ_q and a $(q+1)$-cell σ_{q+1} is an integer that is nonzero only for a finite number of q-cells σ_q and satisfies

$$\sum_{\sigma_q}[\sigma_{q+1} : \sigma_q][\sigma_q : \sigma_{q-1}] = 0 \tag{3.5}$$

for $q \geq 1$. Given $q \geq 0$, we denote by $C_q(K)$ the free abelian group generated by the q-cells, and the elements of $C_q(K)$ are called *q-chains*. The boundary operator on $C_q(K)$ is the homomorphism

$$\partial : C_q(K) \to C_{q-1}(K)$$

of abelian groups given by

$$\partial\sigma_q = \sum_{\sigma_{q-1}}[\sigma_q : \sigma_{q-1}]\sigma_{q-1} \tag{3.6}$$

for each generator σ_q of $C_q(K)$, where the summation is over the generators σ_{q-1} of $C_{q-1}(K)$. Then it can be shown that ∂ satisfies $\partial^2 = \partial \circ \partial = 0$.

Given an abelian group A, we consider the associated group of *q-cochains* given by

$$C^q(K, A) = \mathrm{Hom}(C_q(K), A). \tag{3.7}$$

Since $C_q(K)$ is generated by the q-cells, a q-cochain f is uniquely determined by its values $f(\sigma_q)$ for the q-cells σ_q. The coboundary operator

$$\delta : C^q(K, A) \to C^{q+1}(K, A) \tag{3.8}$$

on $C^q(K, A)$ is defined by

$$(\delta f)(c) = f(\partial c) \tag{3.9}$$

for all $f \in C^q(K, A)$ and $c \in C_{q+1}(K)$, and the condition $\partial^2 = 0$ implies $\delta^2 = 0$. Then the *q-th cohomology group of the complex K over A* is given by the quotient

$$H^q(K, A) = Z^q(K, A)/B^q(K, A),$$

where $Z^q(K, A)$ is the kernel of $\delta : C^q(K, A) \to C^{q+1}(K, A)$ and $B^q(K, A)$ is the image $B^q(K, A)$ of $\delta : C^{q-1}(K, A) \to C^q(K, A)$.

We now assume that a group Γ acts on K and on A, both on the left. Given $q \geq 0$, an element $f \in C^q(K, A)$ is said to be an *equivariant q-cochain* if it satisfies

$$f(\gamma c) = \gamma f(c) \tag{3.10}$$

for all $\gamma \in \Gamma$ and $c \in C_q(K)$, where $C^q(K, A)$ is as in (3.7). We denote by $C^q_E(K, A)$ the subgroup of $C^q(K, A)$ consisting of the equivariant cochains. If δ is the coboundary map in (3.8) and if f is an equivariant q-cochain, then we have

$$\delta f(\gamma c_{q+1}) = f(\partial \gamma c_{q+1}) = f(\gamma \partial c_{q+1}) = \gamma f(\partial c_{q+1}) = \gamma[\delta f(c_{q+1})]$$

for all $\gamma \in \Gamma$, which shows that δf is an equivariant $(q+1)$-cochain. We define an *equivariant q-cocycle* to be an element of the group

$$Z^q_E(K, A) = Z^q(K, A) \cap C^q_E(K, A)$$

and an *equivariant q-coboundary* an element of the subgroup

$$B^q_E(K, A) = \delta C^{q-1}_E(K, A) \tag{3.11}$$

of $B^q(K, A)$. Then the quotient group

$$H^q_E(K, A) = Z^q_E(K, A)/B^q_E(K, A) \tag{3.12}$$

is the *equivariant q-th cohomology group* of K over A.

We denote by $Z^q_R(K, A)$ the subgroup of $C^q(K, A)$ consisting of the cochains with an equivariant coboundary, that is,

$$Z^q_R(K, A) = \{c \in C^q(K, A) \mid \delta c \in B^{q+1}_E(K, A)\}. \tag{3.13}$$

An element of $Z^q_R(K, A)$ is called a *residual q-cocycle*. A *residual q-coboundary*, on the other hand, is an element of the group

$$B^q_R(K, A) = B^q(K, A) + C^q_E(K, A).$$

If $b \in B^q(K,A)$ and $c \in C^q_E(K,A)$, then by (3.11) the element $(b+c) \in B^q_R(K,A)$ satisfies

$$\delta(b+c) = \delta c \in \delta C^q_E(K,A) = B^{q+1}_E(K,A);$$

hence by (3.13) the group $B^q_R(K,A)$ is a subgroup of $Z^q_R(K,A)$. The corresponding quotient group

$$H^q_R(K,A) = Z^q_R(K,A)/B^q_R(K,A)$$

is the *residual q-th cohomology group* of K over A. Then it can be shown (cf. [4]) that there is an exact sequence of the form

$$\cdots \xrightarrow{\delta} H^q_E(K,A) \xrightarrow{\xi} H^q(K,A) \xrightarrow{\eta} H^q_R(K,A) \xrightarrow{\delta} H^{q+1}_E(K,A) \to \cdots, \tag{3.14}$$

where the homomorphisms ξ and η are induced by the inclusions

$$Z^q_E(K,A) \subset Z^q(K,A) \subset Z^q_R(K,A),$$

$$B^q_E(K,A) \subset B^q(K,A) \subset B^q_R(K,A)$$

and the map δ is given by the coboundary map on $C^q(K,A)$.

We now consider the complex K_Γ defined as follows. The q-cells in K_Γ are ordered $(q+1)$-tuples $(\gamma_0,\dots,\gamma_q)$ of elements of Γ, so that $C_q(K_\Gamma)$ is the free abelian group generated by the $(q+1)$-fold Cartesian product Γ^{q+1} of Γ. Given a q-cell $\widetilde{\gamma} = (\gamma_0,\dots,\gamma_q)$ and a $(q-1)$-cell $\widetilde{\alpha} = (\alpha_0,\dots,\alpha_{q-1})$, we define the incidence number $[\widetilde{\gamma}:\widetilde{\alpha}]$ to be $(-1)^i$ if $\widetilde{\alpha} = (\gamma_0,\dots,\widehat{\gamma}_i,\dots,\gamma_q)$ and zero otherwise, where $\widehat{\gamma}_i$ means deleting the entry γ_i. Then it can be shown that the integer $[\widetilde{\gamma}:\widetilde{\alpha}]$ satisfies (3.5), so that K_Γ is indeed a complex. By (3.6) its boundary operator on $C_q(K_\Gamma)$ is given by

$$\partial(\gamma_0,\dots,\gamma_q) = \sum_{i=0}^{q} (-1)^i (\gamma_0,\dots,\widehat{\gamma}_i,\dots,\gamma_q) \in C_{q-1}(K_\Gamma) \tag{3.15}$$

for $\gamma_0,\dots,\gamma_q \in \Gamma$. We define the left action of the group Γ acts on $C_q(K_\Gamma)$ by

$$\gamma(\gamma_0,\dots,\gamma_q) = (\gamma\gamma_0,\dots,\gamma\gamma_q) \tag{3.16}$$

for all $\gamma \in \Gamma$ and $(\gamma_0,\dots,\gamma_q) \in \Gamma^{q+1}$. Thus, if Γ acts on an abelian group A on the left, then we can consider the equivariant cohomology groups $H^q_E(K_\Gamma,A)$ of K_Γ over A.

Proposition 3.2. *Given a left Γ-module A, there is a canonical isomorphism*

$$H^q(\Gamma,A) \cong H^q_E(K_\Gamma,A) \tag{3.17}$$

for each $q \geq 0$.

Proof. For each $q \geq 0$ the group of q-cochains over A associated to the complex K_q is given by

$$C^q(K_\Gamma,A) = \operatorname{Hom}(C^q(K_\Gamma),A).$$

Thus $C^q(K_\Gamma,A)$ consists of maps $f : C_q(K_\Gamma) \to A$ satisfying

$$f\Big(\sum_i m_i \widetilde{\gamma}_i\Big) = \sum_i m_i f(\widetilde{\gamma}_i)$$

where $\widetilde{\gamma}_i$ is a q-cell in K_Γ and $m_i \in A$ for each i. Therefore $C^q(K_\Gamma, A)$ may be regarded as the free abelian group generated by the maps of the form

$$h : \Gamma^{q+1} \to A.$$

By (3.10) and (3.16) an element $f \in C^q(K_\Gamma, A)$ is equivariant if

$$\gamma f(\gamma_0, \dots, \gamma_q) = f(\gamma(\gamma_0, \dots, \gamma_q)) = f(\gamma\gamma_0, \dots, \gamma\gamma_q) \tag{3.18}$$

for each $\gamma \in \Gamma$ and each generator $(\gamma_0, \dots, \gamma_q) \in \Gamma^{q+1}$ of $C_q(K_\Gamma)$. By (3.9) the coboundary map $\delta : C^q(K_\Gamma, A) \to C^{q+1}(K_\Gamma, A)$ is given by

$$\begin{aligned}(\delta f)(\gamma_0, \dots, \gamma_{q+1}) &= f(\partial(\gamma_0, \dots, \gamma_{q+1})) \\ &= f\left(\sum_{i=0}^{q+1} (-1)^i (\gamma_0, \dots, \widehat{\gamma}_i, \dots, \gamma_{q+1})\right) \\ &= \sum_{i=0}^{q+1} (-1)^i f(\gamma_0, \dots, \widehat{\gamma}_i, \dots, \gamma_{q+1})\end{aligned} \tag{3.19}$$

for all $f \in C^q(K_\Gamma, A)$, where we used (3.15). Thus we see that the space of equivariant elements of $C^q(K_\Gamma, A)$ coincides with the space $\mathfrak{C}^q(\Gamma, A)$ of homogeneous q-cochains considered in Section 3.1; hence the proposition follows. □

3.3. **Hecke operators on group cohomology.** In this section, we discuss Hecke operators acting on the group cohomology. Let G be a fixed group. If Γ is a subgroup of G, as in Section 2.2 we denote by $\widetilde{\Gamma}$ its commensurator. Given a subsemigroup Δ of G, recall that $\mathcal{C}(\Delta)$ is the set of mutually commensurable subgroups Γ of G such that

$$\Gamma \subset \Delta \subset \widetilde{\Gamma}.$$

We choose an element $\Gamma \in \mathcal{C}(\Delta)$ and denote by $\mathcal{H}(\Gamma; \Delta)$ the associated Hecke algebra described in Section 2.3. Thus $\mathcal{H}(\Gamma; \Delta)$ is the $\mathbb{Z}$-algebra generated by double cosets $\Gamma\alpha\Gamma$ with $\alpha \in \Delta$.

Given a subgroup Γ of G, we consider the Hecke algebra $\mathcal{H}(\Gamma; \widetilde{\Gamma})$ associated to the subsemigroup $\Delta = \widetilde{\Gamma}$ of G. Let $\Gamma\alpha\Gamma$ with $\alpha \in \widetilde{\Gamma}$ be an element of $\mathcal{H}(\Gamma; \widetilde{\Gamma})$ that has a decomposition of the form

$$\Gamma\alpha\Gamma = \coprod_{i=1}^{d} \Gamma\alpha_i \tag{3.20}$$

for some $\alpha_1, \dots, \alpha_d \in \widetilde{\Gamma}$. Since $\Gamma\alpha\Gamma\gamma = \Gamma\alpha\Gamma$ for each $\gamma \in \Gamma$, we have

$$\Gamma\alpha\Gamma = \coprod_{i=1}^{d} \Gamma\alpha_i = \coprod_{i=1}^{d} \Gamma\alpha_i\gamma$$

for all $\gamma \in \Gamma$. Thus for $1 \le i \le d$, we see that

$$\alpha_i\gamma = \xi_i(\gamma) \cdot \alpha_{i(\gamma)} \tag{3.21}$$

for some element $\xi_i(\gamma) \in \Gamma$, where $(\alpha_{1(\gamma)}, \dots, \alpha_{d(\gamma)})$ is a permutation of $(\alpha_1, \dots, \alpha_d)$. For each i and $\gamma, \gamma' \in \Gamma$ we have

$$(\alpha_i \gamma)\gamma' = \xi_i(\gamma) \cdot \alpha_{i(\gamma)}\gamma' = \xi_i(\gamma) \cdot \xi_{i(\gamma)}(\gamma') \cdot \alpha_{i(\gamma)(\gamma')}.$$

Comparing this with $\alpha_i(\gamma\gamma') = \xi_i(\gamma\gamma')\alpha_{i(\gamma\gamma')}$, we see that

$$i(\gamma\gamma') = i(\gamma)(\gamma'), \quad \xi_i(\gamma\gamma') = \xi_i(\gamma) \cdot \xi_{i(\gamma)}(\gamma') \tag{3.22}$$

for all $\gamma, \gamma' \in \Gamma$.

Given a nonnegative integer p and a Γ-module M, let $\mathfrak{C}^p(\Gamma, M)$ be the group of homogeneous p-cochains described in Section 3.1. For an element $\phi \in \mathfrak{C}^p(\Gamma, M)$ and a double coset $\Gamma\alpha\Gamma$ with $\alpha \in \widetilde{\Gamma}$ that has a decomposition as in (3.20), we consider the map $\phi' : \Gamma^{p+1} \to M$ given by

$$\phi'(\gamma_0, \dots, \gamma_p) = \sum_{i=1}^{d} \alpha_i^{-1} \cdot \phi(\xi_i(\gamma_0), \dots, \xi_i(\gamma_p)),$$

where the maps $\xi_i : \Gamma \to \Gamma$ are determined by (3.21). Then it is known that ϕ' is an element of $\mathfrak{C}^p(\Gamma, M)$ (see [13]). Thus each double coset $\Gamma\alpha\Gamma$ with $\alpha \in \widetilde{\Gamma}$ determines the $\mathbb{C}$-linear map

$$\mathfrak{T}(\alpha) : \mathfrak{C}^p(\Gamma, M) \to \mathfrak{C}^p(\Gamma, M) \tag{3.23}$$

defined by

$$(\mathfrak{T}(\alpha)\phi)(\gamma_0, \dots, \gamma_p) = \sum_{i=1}^{d} \alpha_i^{-1} \cdot \phi(\xi_i(\gamma_0), \dots, \xi_i(\gamma_p)) \tag{3.24}$$

for $\phi \in \mathfrak{C}^p(\Gamma, M)$, where $\Gamma\alpha\Gamma = \coprod_{1 \le i \le d} \Gamma\alpha_i$ and each ξ_i is as in (3.21). Then the map $\mathfrak{T}(\alpha)$ is independent of the choice of representatives of the coset decomposition of $\Gamma\alpha\Gamma$ modulo Γ. Furthermore, it can be shown that

$$\mathfrak{T}(\alpha) \circ \delta_p = \delta_p \circ \mathfrak{T}(\alpha) \tag{3.25}$$

for each $p \ge 1$, where δ_p and δ_{p+1} are coboundary maps on $\mathfrak{C}^p(\Gamma, M)$ and $\mathfrak{C}^{p+1}(\Gamma, M)$, respectively. Thus the map $\mathfrak{T}(\alpha)$ in (3.23) induces a homomorphism

$$\mathfrak{T}(\alpha) : H^p(\Gamma, M) \to H^p(\Gamma, M),$$

which is the *Hecke operator* on $H^p(\Gamma, M)$ corresponding to α.

The Hecke operators can also be described by using nonhomogeneous cochains as follows. For each $q \ge 0$ we denote by $C^q(\Gamma, M)$ the group of nonhomogeneous q-cochains over M as in Section 3.1. Given $f \in C^q(\Gamma, M)$ and $\alpha \in \widetilde{\Gamma}$ with $\Gamma\alpha\Gamma$ as in (3.20), we set

$$\begin{aligned}(T(\alpha)f)&(\gamma_1, \dots, \gamma_q) \\ &= \sum_{i=1}^{d} \alpha_i^{-1} f(\xi_i(\gamma_1), \xi_{i(\gamma_1)}(\gamma_2), \xi_{i(\gamma_1\gamma_2)}(\gamma_3), \dots, \xi_{i(\gamma_1\cdots\gamma_{q-1})}(\gamma_q))\end{aligned} \tag{3.26}$$

for all $\gamma_1, \dots, \gamma_q \in \Gamma$.

Proposition 3.3. *Given $\alpha \in \widetilde{\Gamma}$, the map $T(\alpha)f : \Gamma^q \to M$ is an element of $C^q(\Gamma, M)$ and satisfies*

$$T(\alpha)f = (\mathfrak{T}(\alpha)f_H)_N$$

for all $f \in C^q(\Gamma, M)$, where the operators

$$(\cdot)_H : C^q(G,M) \to \mathfrak{C}^q(G,M), \quad (\cdot)_N : \mathfrak{C}^q(G,M) \to C^q(G,M)$$

are as in (3.3) and (3.4).

Proof. Given $f \in C^q(\Gamma, M)$, by (3.3) we have

$$f_H(\sigma_0, \sigma_1, \dots, \sigma_q) = \sigma_0 \cdot f(\sigma_0^{-1}\sigma_1, \sigma_1^{-1}\sigma_2, \dots, \sigma_{q-1}^{-1}\sigma_q)$$

for all $\sigma_0, \sigma_1, \dots, \sigma_q \in \Gamma$. Thus for $\alpha \in \widetilde{\Gamma}$, using (3.24), we obtain

$$\begin{aligned}(\mathfrak{T}(\alpha)f_H)(\sigma_0, \sigma_1, \dots, \sigma_q) &= \sum_{i=1}^d \alpha_i^{-1} f_H(\xi_i(\sigma_0), \dots, \xi_i(\sigma_q)) \\ &= \sum_{i=1}^d \alpha_i^{-1}\xi_i(\sigma_0)^{-1} f(\xi_i(\sigma_0)^{-1}\xi_i(\sigma_1), \xi_i(\sigma_1)^{-1}\xi_i(\sigma_2), \dots, \xi_i(\sigma_{q-1})^{-1}\xi_i(\sigma_q)).\end{aligned}$$

Hence by using (3.4) we have

$$\begin{aligned}(\mathfrak{T}(\alpha)f_H)_N(\gamma_1, \dots, \gamma_q) &= (\mathfrak{T}(\alpha)f_H)(1, \gamma_1, \gamma_1\gamma_2, \dots, \gamma_1\gamma_2\cdots\gamma_q) \\ &= \sum_{i=1}^d \alpha_i^{-1} f(\xi_i(\gamma_1), \xi_i(\gamma_1)^{-1}\xi_i(\gamma_1\gamma_2), \xi_i(\gamma_1\gamma_2)^{-1}\xi_i(\gamma_1\gamma_2\gamma_3), \dots \\ &\qquad \dots, \xi_i(\gamma_1\cdots\gamma_{q-1})^{-1}\xi_i(\gamma_1\cdots\gamma_{q-1}\gamma_q))\end{aligned}$$

for all $\gamma_1, \dots, \gamma_q \in \Gamma$. However, it follows from (3.22) that

$$\xi_i(\gamma_1\cdots\gamma_{k-1})^{-1}\xi_i(\gamma_1\cdots\gamma_{k-1}\gamma_k) = \xi_{i(\gamma_1\cdots\gamma_{k-1})}(\gamma_k)$$

for $2 \le k \le q$. Hence we obtain

$$(\mathfrak{T}(\alpha)f_H)_N(\gamma_1, \dots, \gamma_q) = \sum_{i=1}^d \alpha_i^{-1} f(\xi_i(\gamma_1), \xi_{i(\gamma_1)}(\gamma_2), \xi_{i(\gamma_1\gamma_2)}(\gamma_3), \dots, \xi_{i(\gamma_1\cdots\gamma_{q-1})}(\gamma_q)),$$

and therefore the proposition follows from this and (3.26). □

Let $\partial_q : C^q(\Gamma, M) \to C^{q+1}(\Gamma, M)$ and $\partial_{q+1} : C^{q+1}(\Gamma, M) \to C^{q+2}(\Gamma, M)$ be the coboundary maps for nonhomogeneous cochains. Then, using Lemma 3.1 and (3.25), we have

$$\begin{aligned}(\partial_{q+1}T(\alpha)f)_H &= \delta_{q+1}(T(\alpha)f)_H = \delta_{q+1}(\mathfrak{T}(\alpha)f_H) \\ &= \mathfrak{T}(\alpha)\delta_q f_H = T(\alpha)(\partial_q f)_H = (T(\alpha)\partial_q f)_H\end{aligned}$$

for all $f \in C^q(\Gamma, M)$; hence it follows that

$$T(\alpha) \circ \partial_q = \partial_{q+1} \circ T(\alpha)$$

for each $q \ge 0$. Therefore the map $T(\alpha) : C^q(\Gamma, M) \to C^q(\Gamma, M)$ also induces the Hecke operator

$$T(\alpha) : H^q(\Gamma, M) \to H^q(\Gamma, M)$$

on $H^q(\Gamma, M)$ that is compatible with $\mathfrak{T}(\alpha)$.

4. De Rham cohomology

The focus of this section is on the de Rham cohomology of differentiable manifolds with coefficients in a vector bundle and Hecke operators on such cohomology. In Section 4.1 we review basic properties of the sheaf cohomology including the sheaf-theoretic interpretation of the de Rham and C^∞ singular cohomology of differentiable manifolds with coefficients in a real vector space. If Γ is a fundamental group of a manifold X and ρ is a representation of Γ in a finite-dimensional real vector space, we can consider the associated vector bundle $\mathcal{V}_\rho$ over X. In Section 4.2 we construct the de Rham cohomology of X with coefficients in $\mathcal{V}_\rho$. This cohomology is identified, in Section 4.3, with certain cohomology of the universal covering space of X associated to the representation ρ of Γ. We use this identification to introduce Hecke operators on the de Rham cohomology of X with coefficients in $\mathcal{V}_\rho$ (cf. [6]).

4.1. **Cohomology of sheaves.** Let X be a topological space, and let $\mathcal{S}$ be a sheaf over X of certain algebraic objects, such as abelian groups, rings, and modules (see e.g. [18] for the definition and basic properties of sheaves). If U is an open subset of X, we denote by $\mathbf{\Gamma}(X, \mathcal{S})$ or $\mathcal{S}(U)$ the space of sections of $\mathcal{S}$ over U. Then a *resolution* of $\mathcal{S}$ is an exact sequence of morphisms of sheaves of the form

$$0 \to \mathcal{S} \to \mathcal{F}^0 \to \mathcal{F}^1 \to \mathcal{F}^2 \to \cdots,$$

which we also write as

$$0 \to \mathcal{S} \to \mathcal{F}^\bullet$$

in terms of the graded sheaf $\mathcal{F}^\bullet = \{\mathcal{F}^i\}_{i \geq 0}$ over X.

Example 4.1. (i) Let A be an abelian group regarded as a constant sheaf over a topological space X. Given an open set $U \subset X$, let $S^p(U, A)$ denote the group of singular p-cochains in U with coefficients in A. If U is a unit ball in a Euclidean space, then its cohomology group is zero. Hence the sequence

$$\cdots \to S^{p-1}(U, A) \xrightarrow{\delta} S^p(U, A) \xrightarrow{\delta} S^{p+1}(U, A)$$

is exact, where δ denotes the usual coboundary operator for singular cochains. We denote by $\mathcal{S}^p(A)$ the sheaf over X generated by the presheaf $U \mapsto S^p(U, A)$. Then the previous exact sequence induces the exact sequence

$$0 \to A \to \mathcal{S}^0(A) \xrightarrow{d} \mathcal{S}^1(A) \xrightarrow{d} \cdots,$$

of sheaves, which is a resolution of the sheaf A over X.

(ii) Let $\mathbb{R}$ be the constant sheaf of real numbers, and let X be a differentiable manifold of real dimension n. We denote by $\mathcal{E}^p$ the sheaf of real-valued p-forms on X. Then we have a sequence of the form

$$0 \to \mathbb{R} \xrightarrow{\iota} \mathcal{E}^0 \xrightarrow{d} \mathcal{E}^1 \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{E}^n \to 0, \tag{4.1}$$

where d is the exterior differentiation operator and ι is the natural inclusion map. Using the Poincaré lemma, it can be shown that the sequence (6.4) is exact and therefore is a resolution of the sheaf $\mathbb{R}$.

(iii) Let X be a complex manifold of complex dimension n, and let $\mathcal{E}^{p,q}$ the sheaf of (p,q)-forms on X. Given p with $0 \le p \le n$, we consider the sequence

$$0 \to \Omega^p \to \mathcal{E}^{p,0} \xrightarrow{\overline{\partial}} \mathcal{E}^{p,1} \xrightarrow{\overline{\partial}} \cdots \xrightarrow{\overline{\partial}} \mathcal{E}^{p,n} \to 0, \tag{4.2}$$

where Ω^p denotes the sheaf of holomorphic p-forms on X that is the kernel of morphism $\overline{\partial} : \mathcal{E}^{p,0} \to \mathcal{E}^{p,1}$. Then the $\overline{\partial}$ Poincaré lemma implies the sequence (4.2) is exact and therefore is a resolution of the sheaf Ω^p.

Given a sheaf $\mathcal{S}$ over a topological space X, in order to define the cohomology of X with coefficients in $\mathcal{S}$ we now construct a particular resolution of $\mathcal{S}$. Let $\mathcal{S}^E$ together with a local homeomorphism $\varpi : \mathcal{S}^E \to X$ be the associated *étale space*, which means that $\mathcal{S}^E$ is a topological space such that $\mathcal{S}$ is isomorphic to the sheaf of sections of ϖ. Let $\mathfrak{C}^0(\mathcal{S})$ be the presheaf defined by

$$\mathfrak{C}^0(\mathcal{S})(U) = \{s : U \to \mathcal{S}^E \mid \varpi \circ s = 1_U\}$$

for each open subset $U \subset X$. Then $\mathfrak{C}^0(\mathcal{S})$ is in fact a sheaf and is known as the *sheaf of discontinuous sections of* $\mathcal{S}$ *over* X, and the natural map $\mathcal{S}(U) \to \mathfrak{C}^0(\mathcal{S})(U)$ determines an injective morphism $\mathcal{S} \to \mathfrak{C}^0(\mathcal{S})$ of sheaves. We set

$$\mathcal{F}^1(\mathcal{S}) = \mathfrak{C}^0(\mathcal{S})/\mathcal{S}, \quad \mathfrak{C}^1(\mathcal{S}) = \mathfrak{C}^0(\mathcal{F}^1(\mathcal{S})),$$

and define inductively

$$\mathcal{F}^i(\mathcal{S}) = \mathfrak{C}^{i-1}(\mathcal{S})/\mathcal{F}^{i-1}(\mathcal{S}), \quad \mathfrak{C}^i(\mathcal{S}) = \mathfrak{C}^0(\mathcal{F}^i(\mathcal{S}))$$

for $i \ge 2$. Then the natural morphisms determine short exact sequences of sheaves over X of the form

$$0 \to \mathcal{S} \to \mathfrak{C}^0(\mathcal{S}) \to \mathfrak{C}^1(\mathcal{S}) \to 0,$$
$$0 \to \mathcal{F}^i(\mathcal{S}) \to \mathfrak{C}^i(\mathcal{S}) \to \mathfrak{C}^{i+1}(\mathcal{S}) \to 0$$

for $i \ge 2$. These sequences induce the long exact sequence

$$0 \to \mathcal{S} \to \mathfrak{C}^0(\mathcal{S}) \to \mathfrak{C}^1(\mathcal{S}) \to \mathfrak{C}^2(\mathcal{S}) \to \cdots ,$$

which is called the *canonical resolution of* $\mathcal{S}$. By taking the global section of each term of this exact sequence we obtain a sequence of the form

$$0 \to \mathbf{\Gamma}(X,\mathcal{S}) \to \mathbf{\Gamma}(X,\mathfrak{C}^0(\mathcal{S})) \to \mathbf{\Gamma}(X,\mathfrak{C}^1(\mathcal{S})) \to \cdots ,$$

which is in fact a cochain complex. For each $i \ge 0$ we set

$$C^i(X,\mathcal{S}) = \mathbf{\Gamma}(X,\mathfrak{C}^i(\mathcal{S})),$$

so that the collection $C^\bullet(X,\mathcal{S}) = \{C^i(X,\mathcal{S})\}_{i\ge 0}$ becomes a cochain complex.

Definition 4.2. Given a sheaf $\mathcal{S}$ over X, the q-th cohomology group of the cochain complex $C^\bullet(X,\mathcal{S})$ is called the q-th *cohomology group of* X *with coefficients in* $\mathcal{S}$ and is denoted by $H^q(X,\mathcal{S})$, that is,

$$H^q(X,\mathcal{S}) = H^q(C^\bullet(X,\mathcal{S})) \tag{4.3}$$

for all $q \geq 0$.

If the coboundary homomorphism $C^i(X, \mathcal{S}) \to C^{i+1}(X, \mathcal{S})$ is denoted by δ^i for $i \geq -1$ with $C^{-1}(X, \mathcal{S}) = 0$, then (4.3) means that

$$H^q(X, \mathcal{S}) = \operatorname{Ker} \delta^i / \operatorname{Im} \delta^{i-1}.$$

In particular we have

$$H^0(X, \mathcal{S}) = \operatorname{Ker} \delta^0 = \mathbf{\Gamma}(X, \mathcal{S}).$$

Definition 4.3. (i) A sheaf $\mathcal{F}$ over a topological space X is *flabby* if for any open set $U \subset X$ the restriction map $\mathcal{F}(X) \to \mathcal{F}(U)$ is surjective.

(ii) A sheaf $\mathcal{F}$ over a topological space X is *soft* if for any closed set $U \subset X$ the restriction map $\mathcal{F}(X) \to \mathcal{F}(U)$ is surjective.

(iii) A sheaf $\mathcal{F}$ of abelian groups over a paracompact Hausdorff space X is *fine* if for any disjoint subsets Y_1 and Y_2 of X there is an automorphism $\phi : \mathcal{F} \to \mathcal{F}$ which induces the zero map on a neighborhood of Y_1 and the identity map on a neighborhood of Y_2.

Theorem 4.4. *Let $\mathcal{S}$ be a sheaf over a paracompact Hausdorff space X. If $\mathcal{S}$ is soft, then*

$$H^q(X, \mathcal{S}) = 0$$

for all $q \geq 1$.

Proof. See [18, Theorem 3.11]. □

Definition 4.5. A resolution of a sheaf $\mathcal{S}$ over X of the form

$$0 \to \mathcal{S} \to \mathcal{A}^0 \to \mathcal{A}^1 \to \mathcal{A}^2 \to \cdots$$

is said to be *acyclic* if $H^j(X, \mathcal{A}^i) = 0$ for all $i \geq 0$ and $j \geq 1$.

Let $\mathcal{S}$ be a sheaf of abelian groups over X, and let

$$0 \to \mathcal{S} \to \mathcal{A}^0 \to \mathcal{A}^1 \to \mathcal{A}^2 \to \cdots \tag{4.4}$$

be a resolution of $\mathcal{S}$. By taking the global section of each term of this exact sequence we obtain a cochain complex of the form

$$0 \to \mathbf{\Gamma}(X, \mathcal{S}) \to \mathbf{\Gamma}(X, \mathcal{A}^0) \to \mathbf{\Gamma}(X, \mathcal{A}^1) \to \mathbf{\Gamma}(X, \mathcal{A}^2) \to \cdots .$$

Thus we can consider the cohomology groups $H^q(\mathbf{\Gamma}(X, \mathcal{A}^\bullet))$ of the cochain complex $\mathbf{\Gamma}(X, \mathcal{A}^\bullet) = \{\mathbf{\Gamma}(X, \mathcal{A}^i)\}_{i \geq 0}$.

Theorem 4.6. *If the resolution (4.4) of the sheaf $\mathcal{S}$ over X is acyclic, then there is a canonical isomorphism*

$$H^q(X, \mathcal{S}) = H^q(\mathbf{\Gamma}(X, \mathcal{A}^\bullet))$$

for all $q \geq 0$.

Proof. See [18, Theorem 3.13]. □

Lemma 4.7. *Let $\mathcal{R}$ be a sheaf of rings over X, and let $\mathcal{M}$ be a sheaf of modules over $\mathcal{R}$. If $\mathcal{R}$ is soft, then $\mathcal{M}$ is soft.*

Proof. Let K be a closed subset of X, and consider an element $s \in \mathcal{M}(K)$. Then s can be extended to a neighborhood U of K. Define an element $h \in \mathcal{R}(K \cup (X - U))$ satisfying $h(x) = 1$ for $x \in K$ and $h(x) = 0$ for $x \in X - U$. Since $\mathcal{R}$ is soft, h can be extended to an element $\widetilde{h} \in \mathcal{R}(X)$. Then $\widetilde{h} \cdot s \in \mathcal{M}(X)$ is an extension of s. □

Let F be a vector space over $\mathbb{R}$, and let $\mathcal{E}^q(F)$ with $q \geq 0$ be the sheaf of F-valued q-forms on a differentiable manifold X. Let $\mathcal{S}^q_\infty(F)$ be the sheaf obtained by modifying $\mathcal{S}^q(A)$ in Example 4.1(i) by using $A = F$ and C^∞ singular q-cochains. We consider the corresponding graded sequences $\mathcal{E}^\bullet = \{\mathcal{E}^i\}_{i \geq 0}$ and $\mathcal{S}^\bullet_\infty(F) = \{\mathcal{S}^i_\infty(F)\}_{i \geq 0}$ of sheaves over X. Then the q-th C^∞ *singular cohomology group* $H^q_\infty(X, F)$ and the q-th *de Rham cohomology group* $H^q_{\mathrm{DR}}(X, F)$ with coefficients in F are defined by

$$H^q_\infty(X, F) = H^q(\mathbf{\Gamma}(X, \mathcal{S}^\bullet_\infty(F)), \quad H^q_{\mathrm{DR}}(X, F) = H^q(\mathbf{\Gamma}(X, \mathcal{E}^\bullet(F)))$$

for each $q \geq 0$. On the other hand, if $\mathcal{E}^{p,\bullet} = \{\mathcal{E}^{p,q}\}_{q \geq 0}$ with $\mathcal{E}^{p,q}$ as in Example 4.1(iii), then the *Dolbeault cohomology group* of X of type (p, q) is defined by

$$H^{(p,q)}(X) = H^q(\mathbf{\Gamma}(X, \mathcal{E}^{p,\bullet}))$$

for $p, q \geq 0$.

Theorem 4.8. *(i) Let F be a vector space over $\mathbb{R}$. If X is a differentiable manifold, then there are canonical isomorphisms*

$$H^q(X, F) \cong H^q_\infty(X, F) \cong H^q_{\mathrm{DR}}(X, F)$$

for all $q \geq 0$, where $H^q(X, F)$ denotes the q-th cohomology group of X with coefficients in the constant sheaf F.

(ii) If X is a complex manifold of complex dimension n, then there is a canonical isomorphism

$$H^{(p,q)}(X) \cong H^q(X, \Omega^p),$$

for all $p, q \geq 0$ with $p + q = 2n$, where Ω^p is the sheaf of holomorphic p-forms on X.

Proof. Given a manifold X, there are resolutions of the constant sheaf F of the form

$$0 \to F \to \mathcal{E}^\bullet(F), \quad 0 \to F \to \mathcal{S}^\bullet_\infty(F).$$

Using the argument of the partition of unity, it can be shown that $\mathcal{S}^0_\infty(F)$ and $\mathcal{E}^0(F)$ are soft sheaves. Since the sheaf $\mathcal{S}^q_\infty(F)$ is a module over $\mathcal{S}^0_\infty(F)$ for each $q \geq 0$, it follows from Lemma 4.7 that $\mathcal{S}^q_\infty(F)$ is soft. Thus, using Theorem 4.4 and Theorem 4.6, we see that

$$H^q(X, F) \cong H^q(\mathbf{\Gamma}(X, \mathcal{S}^\bullet_\infty(F)) = H^q_\infty(X, F).$$

Similarly, each $\mathcal{E}^q(X)$ is soft; hence we have

$$H^q(X, F) \cong H^q(\mathbf{\Gamma}(X, \mathcal{E}^\bullet(F)) = H^q_{\mathrm{DR}}(X, F),$$

which proves (i). As for (ii), we consider the resolution (4.2) of Ω^p and use the fact that the sheaves $\mathcal{E}^{p,q}$ are soft. □

4.2. **De Rham cohomology and vector bundles.** Let X be a manifold, and let $\mathcal{D}$ be the universal covering space of X. Let $\Gamma = \pi_1(X)$ be the fundamental group of X, so that X can be identified with the quotient space $\Gamma\backslash\mathcal{D}$.

Let ρ be a representation of Γ in a finite-dimensional real vector space F, and define an action of Γ on $\mathcal{D} \times F$ by

$$\gamma \cdot (z, v) = (\gamma z, \rho(\gamma)v) \tag{4.5}$$

for all $\gamma \in \Gamma$ and $(z, v) \in \mathcal{D} \times F$. We equip the real vector space F with the Euclidean topology and denote by

$$\mathcal{V}_\rho = \Gamma\backslash\mathcal{D} \times F \tag{4.6}$$

the quotient of $\mathcal{D} \times F$ with respect to the Γ-action in (4.5). Then the natural projection map $\mathrm{pr}_1 : \mathcal{D} \times F \to \mathcal{D}$ induces a surjective map $\pi : \mathcal{V}_\rho \to X$ such that the diagram

$$\begin{array}{ccc} \mathcal{D} \times F & \xrightarrow{\widetilde{\varpi}} & \mathcal{V}_\rho \\ {\scriptstyle \mathrm{pr}_1}\downarrow & & \downarrow{\scriptstyle \pi} \\ \mathcal{D} & \xrightarrow{\varpi} & X \end{array} \tag{4.7}$$

is commutative, where $\widetilde{\varpi}$ and ϖ denote the canonical projection maps. The surjective map π determines the structure of a vector bundle over X on $\mathcal{V}_\rho$ as can be seen in the following proposition.

Proposition 4.9. *The set $\mathcal{V}_\rho$ has the structure of a locally constant vector bundle over $X = \Gamma\backslash D$ with fiber F whose fibration is the map $\pi : \mathcal{V}_\rho \to X$ in (4.7).*

Proof. Let $\{U_\alpha\}_{\alpha \in I}$ be an open cover of X such that the inverse image $\pi^{-1}(U_\alpha)$ of each U_α under ϖ is homeomorphic to U_α. By taking smaller open sets if necessary we may assume that $U_\alpha \cap U_\beta$ is either connected or empty for all $\alpha, \beta \in I$. For each $\alpha \in I$ we select a connected component $\widetilde{U}_\alpha$ of $\pi^{-1}(U_\alpha)$. If $U_\alpha \cap U_\beta \neq \emptyset$, then there exists a unique element $\gamma_{\alpha,\beta} \in \Gamma$ such that

$$\gamma_{\alpha,\beta}\widetilde{U}_\alpha \cap \widetilde{U}_\beta \neq \emptyset. \tag{4.8}$$

We define the map $\psi_\alpha : U_\alpha \times F \to \pi^{-1}(U_\alpha)$ by

$$\psi_\alpha(x, y) = \widetilde{\varpi}(\widetilde{x}, y) \tag{4.9}$$

for all $(x, y) \in U_\alpha \times F$, where $\widetilde{x}$ is the element of $\widetilde{U}_\alpha$ with $\varpi(\widetilde{x}) = x$. Then we see easily that ψ_α is a bijection. We shall now introduce a vector space structure on each fiber $\mathcal{V}_{\rho,x} = \pi^{-1}(x)$ with $x \in X$. Given $x \in U_\alpha \subset X$, we define the map $\psi_{\alpha,x} : F \to \mathcal{V}_{\rho,x}$ by

$$\psi_{\alpha,x}(v) = \psi_\alpha(x, v) \tag{4.10}$$

for all $v \in F$. Then $\psi_{\alpha,x}$ is bijective, and therefore we can define a vector space structure on $\mathcal{V}_{\rho,x}$ by transporting the one on F via the map $\psi_{\alpha,x}$. We need to show that such a structure is independent of x. Let $x \in U_\alpha \cap U_\beta$. If $\widetilde{x}_\alpha \in \widetilde{U}_\alpha$ and

$\widetilde{x}_\beta \in \widetilde{U}_\beta$ are the elements with $\varpi(\widetilde{x}_\alpha) = x = \varpi(\widetilde{x}_\alpha)$. Then from (4.8) we see that $\widetilde{x}_\beta = \gamma_{\alpha,\beta}\widetilde{x}_\alpha$. Using this and the relations (4.5), (4.9) and (4.10), we obtain

$$\begin{aligned}\psi_\alpha(x,v) &= \widetilde{\varpi}(\widetilde{x}_\alpha, v) = \widetilde{\varpi}(\gamma_{\alpha,\beta}^{-1}\widetilde{x}_\beta, v)\\ &= \widetilde{\varpi}(\widetilde{x}_\alpha, \rho(\gamma_{\alpha,\beta})v) = \psi_\alpha(x, \rho(\gamma_{\alpha,\beta})v)\end{aligned}$$

for all $v \in F$. Hence we see that the diagram

$$\begin{array}{ccc} F & \xrightarrow{\rho(\gamma_{\alpha,\beta})} & F \\ {\scriptstyle \psi_\alpha(x,v)}\Big\downarrow & & \Big\downarrow{\scriptstyle \psi_\beta(x,v)} \\ \mathcal{V}_{\rho,x} & =\!=\!= & \mathcal{V}_{\rho,x} \end{array}$$

is commutative, which shows that the vector space structure on $\mathcal{V}_{\rho,x}$ is independent of x. Finally, we note that the map

$$\phi_\alpha = \psi_\alpha^{-1} : \pi^{-1}(U_\alpha) \to U_\alpha \times F$$

can be used as a local trivialization for each $\alpha \in I$. □

Given a positive integer p, we first define a function which assigns to each $x \in X$ an alternating p-linear map

$$\xi_x : T_x(X) \times \cdots \times T_x(X) \to \mathcal{V}_{\rho,x}, \tag{4.11}$$

where $T_x(X)$ denotes the tangent space of X at $x \in X$ and $\mathcal{V}_{\rho,x}$ is the fiber of $\mathcal{V}_\rho$ at x. We then define, for each $\alpha \in I$, the function ξ_α on U_α which associates to each $x \in U_\alpha$ an F-valued alternating p-linear map $\xi_\alpha(x)$ given by

$$\xi_\alpha(x) = \phi_{\alpha,x} \circ \xi_x, \tag{4.12}$$

where $\phi_{\alpha,x} = \phi_\alpha \mid_{\mathcal{V}_{\rho,x}}$.

Definition 4.10. A *$\mathcal{V}_\rho$-valued p-form* on X is a function ξ on X which assigns to each $x \in X$ an alternating p-linear map ξ_x of the form (4.11) such that the function ξ_α in (4.12) is differentiable.

Let $\{U_\alpha\}_{\alpha\in I}$ be an open cover of X. Noting that $\mathcal{V}_\rho$ is locally constant by Proposition 4.9, we denote by $C_{\alpha,\beta} \in GL(F)$ the constant transition function on $U_\alpha \cap U_\beta$ for $\alpha, \beta \in I$. Then a $\mathcal{V}_\rho$-valued p-form on X can be regarded as a collection $\{\omega_\alpha\}_{\alpha\in I}$ of F-valued p-forms ω_α on U_α satisfying

$$\omega_\beta = C_{\alpha,\beta}\omega_\alpha$$

on $U_\alpha \cap U_\beta$ for all $\alpha, \beta \in I$ with $U_\alpha \cap U_\beta \neq \emptyset$. Since each $C_{\alpha,\beta}$ is constant, we have

$$d\omega_\beta = d(C_{\alpha,\beta}\omega_\alpha) = C_{\alpha,\beta}d\omega_\alpha;$$

hence the collection $\{d\omega_\alpha\}_{\alpha\in I}$ determines a $\mathcal{V}(\rho)$-valued $(p+1)$-form on X. Thus, if $\mathcal{E}^p(X, \mathcal{V}_\rho)$ denotes the space of all $\mathcal{V}_\rho$-valued p-forms on X, the map $\{\omega_\alpha\}_{\alpha\in I} \mapsto \{d\omega_\alpha\}_{\alpha\in I}$ determines an operator

$$d : \mathcal{E}^p(X, \mathcal{V}_\rho) \to \mathcal{E}^{p+1}(X, \mathcal{V}_\rho) \tag{4.13}$$

with $d^2 = 0$ for each $p \geq 0$. Then the *de Rham cohomology of X with coefficients in $\mathcal{V}_\rho$* is the cohomology of the cochain complex $\mathcal{E}^\bullet(X, \mathcal{V}_\rho) = \{\mathcal{E}^p(\mathcal{V}(\rho))\}_{p\geq 0}$ with the coboundary operator (4.13). Thus the quotient

$$H^q(X, \mathcal{V}_\rho) = \frac{\operatorname{Ker}(d : \mathcal{E}^q(X, \mathcal{V}_\rho) \to \mathcal{E}^{q+1}(X, \mathcal{V}_\rho))}{d\mathcal{E}^{q-1}(X, \mathcal{V}_\rho)} \tag{4.14}$$

for $q \geq 0$ is the q-th de Rham cohomology of X with coefficients in $\mathcal{V}_\rho$.

4.3. **Hecke operators on de Rham cohomology.** Let $\mathcal{D}$, Γ, $X = \Gamma\backslash\mathcal{D}$, and the representation $\rho : \Gamma \to GL(F)$ be as in Section 4.2. Given $p \geq 0$, the space $\mathcal{E}^p(\mathcal{D}, F)$ of all F-valued p-forms on $\mathcal{D}$ is spanned by the elements of the form $\omega \otimes v$ with $\omega \in \mathcal{E}^p(\mathcal{D})$ and $v \in F$. By setting

$$d(\omega \otimes v) = (d\omega) \otimes v$$

we obtain the map $d : \mathcal{E}^p(\mathcal{D}, F) \to \mathcal{E}^{p+1}(\mathcal{D}, F)$ with $d^2 = 0$; hence we can consider the associated cochain complex $\mathcal{E}^\bullet(\mathcal{D}, F) = \{\mathcal{E}^p(\mathcal{D}, F)\}_{p\geq 0}$ whose cohomology is the de Rham cohomology $H^*_{\mathrm{DR}}(\mathcal{D}, F)$ of $\mathcal{D}$ with coefficients in F. By Theorem 4.8 there is a canonical isomorphism

$$H^q_{\mathrm{DR}}(\mathcal{D}, F) \cong H^q_\infty(\mathcal{D}, F)$$

for each $q \geq 0$. This isomorphism can be described more explicitly as follows. Given $q \geq 0$, the group $\mathcal{S}^q_\infty(\mathcal{D}, F)$ of C^∞ q-cochains considered in Theorem 4.8 can be written as

$$\mathcal{S}^q_\infty(\mathcal{D}, F) = \operatorname{Hom}(\mathcal{S}^\infty_q, F),$$

where $\mathcal{S}^\infty_q$ is the group of C^∞ q-chains. Thus each element of $\mathcal{S}^\infty_q$ is a finite sum of the form $c = \sum_i a_i \Xi_i$ with $a_i \in \mathbb{Z}$, where each $\Xi_i : s \to \mathcal{D}$ is a C^∞ map from a q-simplex in a Euclidean space to $\mathcal{D}$. To each q-form $\omega \in \mathcal{E}^q(\mathcal{D}, F)$ we set

$$f_\omega(c) = \int c\omega = \sum_i a_i \int_s \Xi_i^* \omega \tag{4.15}$$

for $c = \sum_i a_i \Xi_i \in \mathcal{S}^\infty_q$. If $c' = c + \partial c''$ with $c'' \in \mathcal{S}^\infty_{q+1}$, the Stokes theorem implies that

$$f_\omega(c') = \int_{c+\partial c''} \omega = \int_c \omega + \int_{\partial c''} \omega = \int_c \omega + \int_{c''} d\omega = \int_c \omega = f_\omega(c).$$

Thus the map $c \mapsto f_\omega(c)$ is well-defined map on the set of q-cycles in $\mathcal{S}^\infty_q$ and therefore is an element of $\mathcal{S}^q_\infty(\mathcal{D}, F)$. On the other hand, if $\omega' = \omega + d\eta$ with $\eta \in \mathcal{E}^{p+1}(\mathcal{D}, F)$, then we have

$$f_{\omega'}(c) = \int_c (\omega + d\eta) = \int_c \omega = f_\omega(c);$$

hence the map $\omega \mapsto f_\omega$ is a well-defined map from $H^q_{\mathrm{DR}}(\mathcal{D}, F)$ to $H^q_\infty(\mathcal{D}, F)$, and according to Theorem 4.8 this map is an isomorphism.

For each $p \geq 0$, we set

$$\mathcal{E}^p(\mathcal{D}, \Gamma, \rho) = \{\eta \in \mathcal{E}^p(\mathcal{D}, F) \mid \rho(\gamma)\eta = \eta \circ \gamma \text{ for all } \gamma \in \Gamma\}. \tag{4.16}$$

Then we see that

$$d(\mathcal{E}^p(\mathcal{D},\Gamma,\rho)) \subset \mathcal{E}^p(\mathcal{D},\Gamma,\rho);$$

hence we obtain the cochain complex $\mathcal{E}^\bullet(\mathcal{D},\Gamma,\rho) = \{\mathcal{E}^p(\mathcal{D},\Gamma,\rho)\}_{p\geq 0}$. If the q-th cohomology group for this complex is denoted by $H^q(\mathcal{D},\Gamma,\rho)$, then the next proposition shows that it can be identified with the q-th de Rham cohomology group with coefficients in $\mathcal{V}_\rho$.

Proposition 4.11. *There is a canonical isomorphism*

$$H^q(X,\mathcal{V}_\rho) \cong H^q(\mathcal{D},\Gamma,\rho) \tag{4.17}$$

for each $q \geq 0$, where $H^q(X,\mathcal{V}_\rho)$ is as in (4.14).

Proof. Let $\widetilde{\varpi} : \mathcal{D} \times F \to \mathcal{V}_\rho$ and $\varpi : \mathcal{D} \to X$ be the canonical projection maps as in the commutative diagram (4.7). Given $z \in \mathcal{D}$, we define the map $\mu_z : F \to (\mathcal{V}_\rho)_z = \pi^{-1}(\varpi(z))$ by

$$\mu_z(v) = \widetilde{\varpi}(z,v)$$

for all $v \in F$. Then for $\gamma \in \Gamma$ and $v \in F$ we have

$$\mu_{\gamma z}(v) = \widetilde{\varpi}(\gamma z,v) = \widetilde{\varpi}(\gamma^{-1}(\gamma z,v)) = \widetilde{\varpi}(z,\rho(\gamma)^{-1}v) = \mu_z(\rho(\gamma)^{-1}v);$$

hence we see that

$$\mu_{\gamma z}^{-1} = \rho(\gamma)\mu_z^{-1}. \tag{4.18}$$

If $\eta \in \mathcal{E}^q(\mathcal{V}_\rho)$, we define the element $\widetilde{\eta} \in \mathcal{E}^q(\mathcal{D})$ by

$$\widetilde{\eta}_z(u_1,\dots,u_q) = \mu_z^{-1}\eta_{\varpi(z)}(\varpi_* u_1,\dots,\varpi_* u_q)$$

for all $z \in \mathcal{D}$ and $u_1,\dots,u_q \in T_z(\mathcal{D})$. Using this and (4.18), we have

$$\begin{aligned}\widetilde{\eta}_{\gamma z}(\gamma_* u_1,\dots,\gamma_* u_q) &= \mu_{\gamma z}^{-1}\eta_{\varpi(\gamma z)}(\varpi_*\gamma_* u_1,\dots,\varpi_*\gamma_* u_q)\\ &= \rho(\gamma)\mu_z^{-1}\eta_{\varpi(z)}(\varpi_* u_1,\dots,\varpi_* u_q)\\ &= \rho(\gamma)\widetilde{\eta}_z(u_1,\dots,u_q)\end{aligned}$$

for all $\gamma \in \Gamma$, which implies that $\widetilde{\eta} \in \mathcal{E}^q(\mathcal{D},\Gamma,\rho)$. Now we see easily that the map $\eta \mapsto \widetilde{\eta}$ determines an isomorphism between $\mathcal{E}^q(X,\mathcal{V}_\rho)$ and $\mathcal{E}^q(\mathcal{D},\Gamma,\rho)$; hence the lemma follows. □

We now want to introduce Hecke operators on $H^q(\mathcal{D},\Gamma,\rho)$, which by Proposition 4.11 may be regarded as Hecke operators on $H^q(X,\mathcal{V}_\rho)$. Let $\widetilde{\Gamma}$ denote the commensurator of Γ as in Section 3.3, and consider an element $\alpha \in \widetilde{\Gamma} \subset G$ such that the double coset $\Gamma\alpha\Gamma$ has a decomposition of the form

$$\Gamma\alpha\Gamma = \coprod_{i=1}^{d} \Gamma\alpha_i \tag{4.19}$$

for some elements $\alpha_1,\dots,\alpha_d \in \widetilde{\Gamma}$. Given a p-form $\omega \in \mathcal{E}^p(\mathcal{D})$, we denote by $T(\alpha)\omega \in \mathcal{E}^p(\mathcal{D})$ the p-form defined by

$$T(\alpha)\omega = \sum_{i=1}^{d} \rho(\alpha_i)^{-1}\omega \circ \alpha_i. \tag{4.20}$$

Lemma 4.12. *If $\omega \in \mathcal{E}^p(\mathcal{D}, \Gamma, \rho)$, then $T(\alpha)\omega \in \mathcal{E}^p(\mathcal{D}, \Gamma, \rho)$ for each $\alpha \in \widetilde{\Gamma}$.*

Proof. Given an element $\alpha \in \widetilde{\Gamma}$ satisfying (4.19) and $i \in \{1, \dots, d\}$, let $\alpha_{i(\gamma)}$ be an element of G such that

$$\alpha_i \gamma = \xi_i(\gamma) \cdot \alpha_{i(\gamma)}$$

for some element $\xi_i(\gamma) \in \Gamma$ as in (3.21), so that the set $\{\alpha_{1(\gamma)}, \dots, \alpha_{d(\gamma)}\}$ is a permutation of $\{\alpha_1, \dots, \alpha_d\}$. If $\omega \in \mathcal{E}^p(\mathcal{D}, \Gamma, \rho)$, then by (3.21), (4.16) and (4.20) the p-form $T(\alpha)\omega$ satisfies

$$\begin{aligned}
(T(\alpha)\omega) \circ \gamma &= \sum_{i=1}^{d} \rho(\alpha_i)^{-1} \omega \circ (\alpha_i \gamma) \\
&= \sum_{i=1}^{d} \rho(\alpha_i)^{-1} \omega \circ (\xi_i(\gamma) \alpha_{i(\gamma)}) \\
&= \sum_{i=1}^{d} \rho(\xi_i(\gamma) \alpha_{i(\gamma)} \gamma^{-1})^{-1} \omega \circ (\xi_i(\gamma) \alpha_{i(\gamma)}) \\
&= \sum_{i=1}^{d} \rho(\gamma) \rho(\alpha_{i(\gamma)})^{-1} \rho(\xi_i(\gamma))^{-1} \omega \circ \xi_i(\gamma) \circ \alpha_{i(\gamma)} \\
&= \rho(\gamma) \sum_{i=1}^{d} \rho(\alpha_{i(\gamma)})^{-1} \omega \circ \alpha_{i(\gamma)} = \rho(\gamma) T(\alpha)\omega
\end{aligned}$$

for all $\gamma \in \Gamma$; hence it follows that $T(\alpha)\omega \in \mathcal{E}^p(\mathcal{D}, \Gamma, \rho)$. □

By Lemma 4.12 for each $\alpha \in \widetilde{\Gamma}$ there is a linear map

$$T(\alpha) : \mathcal{E}^q(\mathcal{D}, \Gamma, \rho) \to \mathcal{E}^q(\mathcal{D}, \Gamma, \rho).$$

However, since $T(\alpha)$ commutes with d, the same operator induces the operator

$$T(\alpha) : H^q(\mathcal{D}, \Gamma, \rho) \to H^q(\mathcal{D}, \Gamma, \rho) \tag{4.21}$$

on $H^q(\mathcal{D}, \Gamma, \rho)$. Thus, using the canonical isomorphism (4.17), we obtain the operator

$$T(\alpha) : H^q(X, \mathcal{V}_\rho) \to H^q(X, \mathcal{V}_\rho)$$

for each q, which is a *Hecke operator* on $H^q(X, \mathcal{V}_\rho)$ determined by $\alpha \in \widetilde{\Gamma}$.

5. Cohomology with local coefficients

In this section we discuss the cohomology of a topological space X with coefficients in a system of local groups as well as Hecke operators acting on such cohomology. Section 5.1 contains the description of a system of local groups $\mathcal{L}_\rho$ associated to a representation ρ of the fundamental group of X in a finite-dimensional real vector space. When X is a differentiable manifold, we show that the cohomology of X with coefficients in the sheaf of sections of $\mathcal{L}_\rho$ is canonically isomorphic to the de Rham cohomology of the universal covering space $\mathcal{D}$ of X associated to ρ introduced in Section 4.3. In Section 5.2 we discuss the homology and cohomology of X with coefficients in a general system of local groups. We introduce Hecke operators in Section 5.3 acting on de Rham cohomology of X with coefficients in the vector bundle $\mathcal{V}_\rho$ considered in Section 4.2.

5.1. **Local systems.** Let X be an arcwise connected topological space with fundamental group $\Gamma = \pi_1(X)$, and let $\mathcal{D}$ be its universal covering space. Thus X can be identified with the quotient space $\Gamma\backslash\mathcal{D}$. Given $x, y \in X$, we denote by $\alpha_{xy} \in \Gamma$ the homotopy class of curves from x to y. The homotopy class containing the inverse of a curve belonging to α_{xy} is denoted by α_{xy}^{-1}, and the symbol $\alpha_{xy}\alpha_{yz} \in \Gamma$ denotes the homotopy class obtained by traversing first a path in the class α_{xy} followed by a path in the class α_{yz}. We fix a base point $x_0 \in X$, and denote the class α_{x_0x} simply by α_x. We also use α to denote the class $\alpha_{x_0x_0}$ of closed paths.

Definition 5.1. A *system of local groups* on X is a collection $\widetilde{A} = \{A_x\}_{x\in X}$ of groups A_x for $x \in X$ satisfying the following conditions:

(i) For each class α_{xy} of paths in X there is an isomorphism $A_x \to A_y$.

(ii) If the transform of $a \in A_x$ under the isomorphism in (i) is denoted by $a\alpha_{xy} \in A_y$, then we have $(a\alpha_{xy})\alpha_{yz} = a(\alpha_{xy}\alpha_{yz})$ for all $x, y, z \in X$ and $a \in A_x$.

The group A_{x_0}, where x_0 is the base point of X, will be denoted simply by A. Then each element $\alpha \in \Gamma = \pi_1(X)$ determines an endomorphism $a \mapsto a\alpha = \alpha_{x_0x_0}$ of A; hence Γ acts on A on the right.

Let ρ be a representation of Γ in a finite-dimensional real vector space F. We denote by F_d the vector space F equipped with the discrete topology, and set

$$\mathcal{L}_\rho = \Gamma\backslash\mathcal{D} \times F_d$$

where the quotient is taken with respect to the action in (4.5) with F replaced with F_d. Then the natural projection map $\mathcal{D} \times F_d \to \mathcal{D}$ induces a surjective map $\pi : \mathcal{L}_\rho \to X = \Gamma\backslash\mathcal{D}$.

Proposition 5.2. *For each $x \in X$, let $\mathcal{L}_{\rho,x} = \pi^{-1}(x)$ be the fiber of $\mathcal{L}_\rho$ over x. Then the space $\mathcal{L}_\rho$, regarded as the collection $\{\mathcal{L}_{\rho,x}\}_{x\in X}$ of its fibers is a system of local groups on X.*

Proof. For each $x \in X$ the fiber $\mathcal{L}_{\rho,x}$ of $\mathcal{L}_\rho$ over x is isomorphic to the discrete additive group F_d. There exist an open covering $\{U_\alpha\}_{\alpha\in A}$ of X and a homeomorphism

$$\psi_\alpha : U_\alpha \times F_d \to \pi^{-1}(U_\alpha) \tag{5.1}$$

for each $\alpha \in A$ such that $\psi_\alpha(z, v) = \pi(z)$ for all $(z, v) \in U_\alpha \times F_d$ and ψ_α induces an isomorphism $\{z\} \times F_d \cong \mathcal{L}_{\rho,\pi(z)}$ for each $z \in U_\alpha$. If $x, y \in X$, since F_d is totally disconnected, any curve α_{xy} from x to y determines uniquely an isomorphism $\mathcal{L}_{\rho,x} \cong \mathcal{L}_{\rho,y}$ which depends only on the homotopy class of α_{xy} (see [16, Section 13]). Thus the collection $\{\mathcal{L}_{\rho,x}\}_{x\in X}$ is a system of local groups on X. □

We now assume that X is a differentiable manifold and denote by $\mathcal{V}_\rho = \Gamma\backslash\mathcal{D}\times F$ the vector bundle over X given by (4.6), where F is equipped with the Euclidean topology. We denote by $\widetilde{\mathcal{E}}^p(X, \mathcal{V}_\rho)$ the sheaf of germs of $\mathcal{V}_\rho$-valued p-forms on X.

If $\mathbf{\Gamma}(\widetilde{\mathcal{E}}^p(X, \mathcal{V}_\rho))$ denotes the space of sections of $\widetilde{\mathcal{E}}^p(X, \mathcal{V}_\rho)$, we obtain the cochain complex $\mathbf{\Gamma}(\widetilde{\mathcal{E}}^\bullet(X, \mathcal{V}_\rho)) = \{\mathbf{\Gamma}(\widetilde{\mathcal{E}}^p(X, \mathcal{V}_\rho))\}_{p \geq 0}$ whose coboundary map

$$d : \mathbf{\Gamma}(\widetilde{\mathcal{E}}^p(X, \mathcal{V}_\rho)) \to \mathbf{\Gamma}(\widetilde{\mathcal{E}}^{p+1}(X, \mathcal{V}_\rho))$$

is induced by the exterior differentiation map. Since the natural isomorphism

$$\mathbf{\Gamma}(\widetilde{\mathcal{E}}^p(X, \mathcal{V}_\rho)) \cong \mathcal{E}^p(X, \mathcal{V}_\rho)$$

commutes with d, it determines a canonical isomorphism

$$H^p(\mathbf{\Gamma}(\widetilde{\mathcal{E}}^\bullet(X, \mathcal{V}_\rho)) \cong H^p(X, \mathcal{V}_\rho) \tag{5.2}$$

for each $p \geq 0$.

Proposition 5.3. *Let $\widetilde{\mathcal{L}}_\rho$ be the sheaf of germs of continuous sections of the local system $\mathcal{L}_\rho$ in Proposition 5.2. Then for each $q \geq 0$ there are canonical isomorphisms*

$$H^q(X, \widetilde{\mathcal{L}}_\rho) \cong H^q(X, \mathcal{V}_\rho) \cong H^q(\mathcal{D}, \Gamma, \rho)$$

between the q-th cohomology group $H^q(X, \mathcal{V}_\rho)$ of the complex $\mathcal{E}^\bullet(X, \mathcal{V}_\rho)$ and the q-th cohomology group $H^q(X, \widetilde{\mathcal{L}}_\rho)$ of X with coefficients in $\widetilde{\mathcal{L}}_\rho$.

Proof. The second isomorphism was proved in Proposition 4.11. As for the first isomorphism, by using the Poincaré lemma it can be shown that the sheaf $\widetilde{\mathcal{L}}_\rho$ is locally constant and that the sequence

$$0 \to \widetilde{\mathcal{L}}_\rho \to \widetilde{\mathcal{E}}^0(X, \mathcal{V}_\rho) \xrightarrow{d} \widetilde{\mathcal{E}}^1(X, \mathcal{V}_\rho) \xrightarrow{d} \cdots$$

is exact. Hence by Theorem 4.6 there is a canonical isomorphism

$$H^q(X, \widetilde{\mathcal{L}}_\rho) \cong H^q(\mathbf{\Gamma}(\widetilde{\mathcal{E}}^\bullet(X, \mathcal{V}_\rho)).$$

Thus the lemma follows from this and (5.2). □

5.2. **Homology and cohomology with local coefficients.** Let X, $\mathcal{D}$ and $\Gamma = \pi_1(X)$ be as in Section 5.1, so that X can be identified with the quotient space $\Gamma \backslash \mathcal{D}$. We consider a local system $\widetilde{A} = \{A_x\}$ on X. If $s = \langle p_0, \dots, p_q \rangle$ is a Euclidean simplex and $\eta : s \to X$ is a singular q-simplex in X, we set $A_\eta = A_{\eta(p_0)}$. Since the leading vertex of the i-th face $\eta^{(i)}$ for $1 \leq i \leq q$ coincides with that of η, we see that $A_\eta = A_{\eta^{(i)}}$. For $i = 0$, however, the 0-th face $\eta^{(0)}$ has $\eta(p_1)$ as its leading vertex and is not connected with $\eta(p_0)$. In this case the leading edge

$$\lambda_\eta = \eta(\overline{p_0 p_1}) \tag{5.3}$$

of η is a path in X from $\eta(p_0)$ to $\eta(p_1)$ and yields an isomorphism $a \mapsto a\lambda_\eta$ of $A_\eta = A_{\eta(p_0)}$ onto $A_{\eta(p_1)} = A_{\eta^{(0)}}$.

Given $x, y \in X$ and a path α_{xy} from x to y, we define an isomorphism $\alpha_{xy} : A_y \to A_x$ by

$$\alpha_{xy}(a) = a\alpha_{xy}^{-1} \tag{5.4}$$

for all $a \in A_y$. We assume that the groups A_x are topological and that the isomorphisms $a \mapsto \alpha_{xy}(a)$ of A_y onto A_x are continuous.

We now introduce a cochain complex $C^\bullet(X, \widetilde{A}) = \{C^q(X, \widetilde{A})\}_{q\geq 0}$ defined as follows. Given $q \geq 0$, a q-cochain on X over $\widetilde{A}$ belonging to $C^q(X, \widetilde{A})$ is a function f which assigns an element $f(\eta) \in A_\eta$ to each singular q-simplex η in X. We define the homomorphism $\delta : C^q(X, \widetilde{A}) \to C^{q+1}(X, \widetilde{A})$ by

$$(\delta f)(\eta) = \lambda_\eta(f(\eta^{(0)})) + \sum_{i=1}^{q+1} (-1)^i f(\eta^{(i)}) \tag{5.5}$$

for each $(q+1)$-simplex η and $f \in C^q(X, \widetilde{A})$, where λ_η is as in (5.3). Then it can be shown that the homomorphism δ satisfies $\delta^2 = 0$ and therefore is a coboundary map for the cochain complex $C^\bullet(X, \widetilde{A})$. Thus we obtain the associated q-th cohomology group

$$H^q(X, \widetilde{A}) = \frac{\operatorname{Ker}(\delta : C^q(X, \widetilde{A}) \to C^{q+1}(X, \widetilde{A}))}{\delta C^q(X, \widetilde{A})}$$

of X with coefficients in $\widetilde{A}$.

Let $x_0 \in X$ be a base point, so that the fundamental group of X can be written as $\Gamma = \pi_1(X, x_0)$, and set $A = A_{x_0} \in \widetilde{A}$. Then A is an abelian group, and by (5.4) the group Γ acts on A on the left. Let $K_\mathcal{D}$ be the singular complex in $\mathcal{D}$, and for each $q \geq 0$ let $C_q(\mathcal{D}) = C_q(K_\mathcal{D})$ denote the group of singular q-chains in $\mathcal{D}$. Then $C_q(\mathcal{D})$ is the free abelian group generated by the singular q-simplexes in $\mathcal{D}$, and there is a boundary map $\partial_\mathcal{D} : C_q(\mathcal{D}) \to C_{q-1}(\mathcal{D})$ given by

$$\partial_\mathcal{D} \eta\langle p_0, \dots, p_q\rangle = \sum_{i=0}^{q} (-1)^i \eta\langle p_0, \dots, \widehat{p}_i, \dots, p_q\rangle$$

for a singular q-simplex η in $\mathcal{D}$ associated to a Euclidean singular q-simplex $\langle p_0, \dots, p_q\rangle$. Then the group of singular q-cochains with coefficients in A is given by

$$C^q(\mathcal{D}, A) = C^q(K_\mathcal{D}, A) = \operatorname{Hom}(C_q(\mathcal{D}), A),$$

and its coboundary operator $\delta_\mathcal{D}$ is defined by

$$\delta_\mathcal{D} f \eta = f \partial_\mathcal{D} \eta$$

for all $f \in C^q(K_\mathcal{D}, A)$ and $\eta \in C^q(K_\mathcal{D})$. By (3.10) a q-cochain $f \in C^q(\mathcal{D}, A)$ is equivariant with respect to Γ if

$$f(\gamma c) = \gamma f(c)$$

for all $\gamma \in \Gamma$ and $c \in C_q(\mathcal{D}, A)$. We denote by $C^q_E(\mathcal{D}, A)$ the subgroup of $C^q(\mathcal{D}, A)$ consisting of the equivariant cochains. If $\delta_\mathcal{D}$ is the coboundary map, by (3.12) the equivariant q-th singular cohomology group of $\mathcal{D}$ over A is given by

$$H^q_E(\mathcal{D}, A) = \frac{(Z^q(\mathcal{D}, A) \cap C^q_E(\mathcal{D}, A))}{\delta_\mathcal{D} C^{q-1}_E(\mathcal{D}, A)},$$

where $Z^q(\mathcal{D}, A)$ denotes the kernel of the map $\delta_\mathcal{D} : C^q(\mathcal{D}, A) \to C^{q+1}(\mathcal{D}, A)$.

Theorem 5.4. *There is a canonical isomorphism*

$$H^q(X, \widetilde{A}) \cong H^q_E(\mathcal{D}, A)$$

for each $q \geq 0$.

Proof. In this proof we shall regard an element $z \in \mathcal{D}$ as the homotopy class of paths in X joining the base point x_0 with πz, where $\pi : \mathcal{D} \to X$ is the natural projection map. Then for each $a \in A_{\pi z}$ the element $za = az^{-1} \in A_{x_0} = A$ is well-defined, and we have

$$z(\alpha a) = (z\alpha)a$$

for each $\alpha \in \Gamma$. If $f \in C^q(X, \widetilde{A})$, we define the cochain $\pi_E f \in C^q(\mathcal{D}, A)$ by

$$\pi_E f(\eta) = z f(\pi\eta) \tag{5.6}$$

for all q-simplex η in $\mathcal{D}$, where $z \in \mathcal{D}$ is the leading vertex of η. Since $f(\pi\eta) \in A_{\pi\eta} = A_{\pi z}$, the element $zf(\pi\eta)$ belongs to A, and therefore $\pi_E f$ is a cochain belonging to $C^q(\mathcal{D}, A)$. Thus (5.6) determines a homomorphism $\pi_E : C^q(X, \widetilde{A}) \to C^q(\mathcal{D}, A)$. If ∂ denotes the coboundary map for $C^q(\mathcal{D}, A)$, then by using (5.6) we have

$$\partial(\pi_E f)(\eta) = \sum_{i=0}^{q+1} (-1)^i \pi_E f(\eta^{(i)}) = \sum_{i=0}^{q+1} (-1)^i z^{(i)} f(\pi\eta^{(i)}),$$

where $\eta^{(i)}$ is the i-th face of η and $z^{(i)} \in \mathcal{D}$ is the leading vertex of $\eta^{(i)}$. Since z is the leading vertex of η, for $1 \leq i \leq q+1$ we see that $z^{(i)} = z$. For $i = 0$, however, we have $z^{(0)} f(\pi\eta^{(0)}) = z\lambda_{\pi\eta} f(\pi\eta^{(0)})$, where $\lambda_{\pi\eta}$ is the leading edge of the simplex $\pi\eta$ in $\mathcal{D}$. Hence we obtain

$$\begin{aligned}
\partial(\pi_E f)(\eta) &= z\lambda_{\pi\eta} f(\pi\eta^{(0)}) + \sum_{i=1}^{q+1} (-1)^i z f(\pi\eta^{(i)}) \\
&= z\left(\lambda_{\pi\eta} f(\pi\eta^{(0)}) + \sum_{i=1}^{q+1} (-1)^i f(\pi\eta^{(i)})\right) \\
&= z(\delta f)(\pi\eta) = (\pi_E \delta f)(\eta),
\end{aligned}$$

where we used (5.5) and (5.6). Thus we see that $\partial\pi_E = \pi_E\delta$. We shall now show that the map π_E is an isomorphism. First, if f is a nonzero element of $C^q(X, \widetilde{A})$, then $f(\eta) \neq 0$ for some q-simplex η in X. Hence there is a simplex $\widetilde{\eta}$ in $\mathcal{D}$ such that $\pi\widetilde{\eta} = \eta$ and $\pi_E f(\widetilde{\eta}) = zf(\eta) \neq 0$, and therefore π_E is injective. To consider the surjectivity of π_E we note that, if η is a q-simplex in $\mathcal{D}$ with leading vertex z and if $\gamma \in \Gamma$, then γz is the leading vertex of $\gamma\eta$ and

$$\pi_E f(\gamma\eta) = \gamma z f(\pi\gamma\eta) = \gamma(zf(\pi\eta)) = \gamma\pi_E f(\eta);$$

hence the cochain $\pi_E f$ is equivariant. Now, let h be an equivariant q-cochain in X over G. Given a q-simplex η in X, we choose a q-simplex $\widetilde{\eta}$ in $\mathcal{D}$ with $\pi\widetilde{\eta} = \eta$ and consider the element $z^{-1}h(\widetilde{\eta})$ of A_η, where z is the leading vertex of $\widetilde{\eta}$. If $\widetilde{\eta}$ is replaced by $\gamma\widetilde{\eta}$ with $\gamma \in \Gamma$, then we have

$$(\gamma z)^{-1} h(\gamma\widetilde{\eta}) = z^{-1}\gamma^{-1}\gamma h(\widetilde{\eta}) = z^{-1}h(\widetilde{\eta}),$$

where we used the fact that h is equivariant. Hence the element $z^{-1}h(\widetilde{\eta})$ of A_η is independent of the choice of $\widetilde{\eta}$. We now define a cochain $f \in C^q(X, \widetilde{A})$ by

$$f(\eta) = z^{-1}h(\widetilde{\eta}).$$

Then we see that

$$(\pi_E f)(\eta) = zf(\pi\widetilde{\eta}) = zf(\eta) = zz^{-1}h(\eta) = h(\eta),$$

and therefore $\pi_E f = h$, which implies the surjectivity of π_E. We have thus shown that π is an isomorphic mapping of the group of cochains $C^q(X, \widetilde{A})$ onto the group $C^q_E(\mathcal{D}, A)$ of equivariant cochains. Since in addition $\partial\pi_E = \pi_E\delta$, it follows that $Z^q(X, \widetilde{A})$ and $B^q(X, \widetilde{A})$ are mapped isomorphically onto the groups $Z^q_E(\mathcal{D}, A)$ and $B^q_E(\mathcal{D}, A)$, respectively. This proves that the map π_E determines the isomorphism $H^q(X, \widetilde{A}) \cong H^q_E(\mathcal{D}, A)$. □

5.3. **Hecke operators.** Let $\mathcal{D}$, Γ and $X = \Gamma\backslash\mathcal{D}$ be as in Section 5.1, and let $\mathcal{V}_\rho$ be the vector bundle over X given by (4.6) associated to a representation $\rho : \Gamma \to GL(F)$ of Γ in a finite-dimensional vector space F over $\mathbb{R}$.

We consider another manifold $X' = \Gamma'\backslash\mathcal{D}'$, where Γ' is the fundamental group and $\mathcal{D}'$ is the universal covering space of X'. Let $\sigma : \Gamma' \to \Gamma$ be a group homomorphism, and let $\widetilde{\tau} : \mathcal{D}' \to \mathcal{D}$ be a C^∞ map that is equivariant with respect to σ, which means that

$$\widetilde{\tau}(\gamma' z') = \sigma(\gamma')\widetilde{\tau}(z')$$

for all $\gamma' \in \Gamma'$ and $z' \in \mathcal{D}'$. Then $\widetilde{\tau}$ induces a C^∞ map $\tau : X' \to X$. We now define an action of Γ' on $\mathcal{D}' \times F$ by

$$\gamma' \cdot (z', v) = (\gamma' z', \rho(\sigma(\gamma'))v)$$

for all $\gamma' \in \Gamma'$, $z' \in \mathcal{D}'$ and $v \in F$. Then the corresponding quotient space

$$\mathcal{V}'_{\rho\circ\sigma} = \Gamma'\backslash\mathcal{D}' \times F \tag{5.7}$$

is a vector bundle over X' with fiber F, whose fibration $\pi' : \mathcal{V}'_{\rho\circ\sigma} \to X'$ is induced by the natural projection map $\mathcal{D}' \times F \to \mathcal{D}'$. The next lemma shows that this bundle is essentially the same as the vector bundle $\tau^*\mathcal{V}_\rho$ over X' obtained by pulling $\mathcal{V}_\rho$ back via τ.

Lemma 5.5. *The bundle $\mathcal{V}'_{\rho\circ\sigma}$ over X' in (5.7) is canonically isomorphic to the pullback bundle $\tau^*\mathcal{V}_\rho$.*

Proof. We note that the pullback bundle $\tau^*\mathcal{V}_\rho$ over X' is given by

$$\tau^*\mathcal{V}_\rho = \{(x', \xi) \in X' \times \mathcal{V}_\rho \mid \tau(x') = \pi(\xi)\}, \tag{5.8}$$

where $\pi : \mathcal{V}_\rho \to X$ is the fibration for the bundle $\mathcal{V}_\rho$. We introduce the notations

$$p : \mathcal{D} \to \Gamma\backslash\mathcal{D} = X, \quad q : \mathcal{D} \times V \to \Gamma\backslash\mathcal{D} \times V = \mathcal{V}_\rho,$$
$$p' : \mathcal{D}' \to \Gamma'\backslash\mathcal{D}' = X', \quad q' : \mathcal{D}' \times V \to \Gamma'\backslash\mathcal{D}' \times V = \mathcal{V}'_{\rho\circ\sigma}$$

for the respective natural projection maps. Then (5.8) can be written in the form

$$\tau^*\mathcal{V}_\rho = \{(p'(z'), q(z, v)) \mid z' \in \mathcal{D}', z \in \mathcal{D}, v \in F, \tau(p'(z')) = \pi(q(z, v))\}.$$

Since $\tau(p'(z')) = p(\widetilde{\tau}(z'))$ and $\pi(q(z,v)) = p(z)$, the condition $\tau(p'(z')) = \pi(q(z,v))$ is equivalent to the relation $z = \gamma\widetilde{\tau}(z')$ for some $\gamma \in \Gamma$. Using this and the fact that $q(\gamma\widetilde{\tau}(z'), v) = q(\widetilde{\tau}(z'), \rho(\gamma)^{-1}v)$, we see that

$$\tau^*\mathcal{V}_\rho = \{(p'(z'), q(\widetilde{\tau}(z'), v)) \mid z' \in \mathcal{D}',\ v \in F\}.$$

Thus we may define a map $\phi : \mathcal{V}'_{\rho\circ\sigma} \to \tau^*\mathcal{V}_\rho$ by

$$\phi(q'(z', v)) = (p'(z'), q(\widetilde{\tau}(z'), v))$$

for all $z' \in \mathcal{D}'$ and $v \in F$. If $\gamma' \in \Gamma'$, we have

$$\begin{aligned}
\phi(q'(\gamma' z', \rho(\sigma(\gamma'))v)) &= (p'(\gamma' z'), q(\widetilde{\tau}(\gamma' z'), \rho(\sigma(\gamma'))v)) \\
&= (p'(\gamma' z'), q(\sigma(\gamma')\widetilde{\tau}(z'), \rho(\sigma(\gamma'))v)) \\
&= (p'(\gamma' z'), q(\widetilde{\tau}(z'), v)) = \phi(q'(z', v));
\end{aligned}$$

hence ϕ is a well-defined surjective map. To verify the injectivity of ϕ we consider elements $z', z'_1 \in \mathcal{D}'$ and $v, v_1 \in F$ satisfying

$$(p'(z'), q(\widetilde{\tau}(z'), v)) = (p'(z'_1), q(\widetilde{\tau}(z'_1), v_1)).$$

Then we have

$$z'_1 = \gamma' z', \quad \widetilde{\tau}(z'_1) = \widetilde{\tau}(\gamma' z') = \sigma(\gamma')\widetilde{\tau}(z'), \quad v_1 = \rho(\sigma(\gamma'))v$$

for some $\gamma' \in \Gamma'$. Thus we obtain

$$q'(z'_1, v_1) = q'(\gamma' z', \rho(\sigma(\gamma'))v) = q'(z', v),$$

and therefore ϕ is injective and the proof of the lemma is complete. □

Let $H^r(X, \mathcal{V}_\rho)$ be the r-th cohomology group for the cochain complex $\{\mathcal{E}^\bullet(X, \mathcal{V}_\rho)\}$ for each $r \geq 0$ considered in Section 4.2. If $H^r(X, \mathcal{V}_\rho)^*$ denotes the dual space of $H^r(X, \mathcal{V}_\rho)$, then we have the natural identification

$$H^r(X, \mathcal{V}_\rho)^* = H^r(X, \mathcal{V}_{\rho^*}),$$

where ρ^* denotes the contragredient of ρ.

We assume that $\tau : X' \to X$ is a smooth ℓ-sheeted covering map for some positive integer ℓ. Then the associated pull-back map $\tau^* : \mathcal{E}^r(X, \mathcal{V}_\rho) \to \mathcal{E}^r(X', \mathcal{V}'_{\rho\circ\sigma})$ determines the homomorphism

$$\tau^* : H^r(X, \mathcal{V}_\rho) \to H^r(X', \mathcal{V}'_{\rho\circ\sigma})$$

of cohomology groups. On the other hand, according to the Poincaré duality, there are canonical isomorphisms

$$P : H^r(X, \mathcal{V}_\rho) \to H^{n-r}(X, \mathcal{V}_\rho)^*,$$

$$P' : H^r(X', \mathcal{V}'_{\rho\circ\sigma}) \to H^{n-r}(X', \mathcal{V}'_{\rho\circ\sigma})^*,$$

where $(\cdot)^*$ denotes the dual space of $(\cdot)$. Then the *Gysin map* associated to τ is the linear map

$$\tau_! : H^r(X', \mathcal{V}'_{\rho\circ\sigma}) \to H^r(X, \mathcal{V}_\rho)$$

such that the diagram

$$\begin{array}{ccc} H^r(X', \mathcal{V}'_{\rho\circ\sigma}) & \xrightarrow{\tau_!} & H^r(X, \mathcal{V}_\rho) \\ {\scriptstyle P'}\downarrow & & \downarrow{\scriptstyle P} \\ H^{n-r}(X', \mathcal{V}'_{\rho\circ\sigma})^* & \xrightarrow{{}^t\tau^*} & H^{n-r}(X, \mathcal{V}_\rho)^* \end{array}$$

is commutative; here ${}^t\tau^*$ is the dual of the linear map

$$\tau^* : H^{n-r}(X, \mathcal{V}_\rho) \to H^{n-r}(X', \mathcal{V}'_{\rho\circ\sigma}).$$

Thus the Gysin map is characterized by the condition

$$\int_X \tau_!(\omega) \wedge \xi = \int_{X'} \omega \wedge \tau^*(\xi) \tag{5.9}$$

for all $\omega \in \mathcal{E}^r(X', \mathcal{V}'_{\rho\circ\sigma})$ and $\xi \in \mathcal{E}^{n-r}(X, \mathcal{V}_{\rho^*})$. In order to discuss Hecke operators on $H^r(X, \mathcal{V}_\rho)$ we now consider a pair (τ, μ) of smooth ℓ-sheeted covering maps $\tau, \mu : X' \to X$.

Definition 5.6. For $0 \le r \le n$ the *Hecke operator on* $H^r(X, \mathcal{V}_\rho)$ associated to the pair (τ, μ) is the map

$$T^r(\tau, \mu) : H^r(X, \mathcal{V}_\rho) \to H^r(X, \mathcal{V}_\rho)$$

given by

$$T^r(\tau, \mu) = \mu_! \circ \tau^*. \tag{5.10}$$

We now consider the case where $\mathcal{D}'$ is equal to $\mathcal{D}$ and Γ' is a subgroup of Γ with $[\Gamma : \Gamma'] = \ell$. Let $\{\varepsilon_1, \dots, \varepsilon_\ell\}$ be the set of coset representatives of Γ' in Γ, so that we have

$$\Gamma = \coprod_{i=1}^{\ell} \Gamma' \varepsilon_i. \tag{5.11}$$

If $\gamma \in \Gamma$, we shall use the same symbol to denote either the map

$$\gamma : \mathcal{D} \to \mathcal{D}$$

sending $z \in \mathcal{D}$ to $\gamma z \in \mathcal{D}$ or the map

$$\gamma : X' \to X$$

which it induces.

Theorem 5.7. *Let $\tau, \mu : X' \to X$ be covering maps, and assume that μ is induced by the identity map on $\mathcal{D}$. Then for $1 \le r \le n$ the associated Hecke operator on $H^r(X, \mathcal{V}_\rho)$ is given by*

$$T^r(\tau, \mu)\omega = \sum_{i=1}^{\ell} \rho(\varepsilon_i)^{-1} (\tau \circ \varepsilon_i)^* \omega$$

for all $\mathcal{E}^r(X, \mathcal{V}_\rho)$.

Proof. First, we shall determine $\mu_! \eta$ for $\eta \in \mathcal{E}^r(X', \mathcal{V}'_{\rho\circ\sigma})$, where $\sigma : \Gamma' \to \Gamma$ is the inclusion map. By (5.9) we have

$$\int_{X'} \eta \wedge \mu^*(\xi) = \int_X \mu_!(\eta) \wedge \xi$$

for all $\xi \in \mathcal{E}^{n-r}(X, \mathcal{V}_{\rho^*})$. Let $\widetilde{\xi} \in \mathcal{E}^{n-r}(\mathcal{D}, \Gamma, \rho^*)$ and $\widetilde{\eta} \in \mathcal{E}^r(\mathcal{D}, \Gamma', \rho \circ \sigma)$ with $\rho \circ \sigma = \rho\,|_{\Gamma'}$ be the differential forms on $\mathcal{D}$ corresponding to ξ and η, respectively. If $\mathcal{F}$ and $\mathcal{F}'$ are fundamental domains of Γ and Γ', respectively, then by (5.11) the domain $\mathcal{F}'$ can be written as a disjoint union of the form

$$\mathcal{F}' = \coprod_{i=1}^{\ell} \varepsilon_i \mathcal{F}.$$

Using this and the fact that the lifting of μ is the identity map on $\mathcal{D}$, we have

$$\begin{aligned}\int_{X'} \eta \wedge \mu^*(\xi) &= \int_{\mathcal{F}'} \widetilde{\eta} \wedge \widetilde{\xi} = \sum_{i=1}^{\ell} \int_{\varepsilon_i F} \widetilde{\eta} \wedge \widetilde{\xi} = \sum_{i=1}^{\ell} \int_F \widetilde{\eta} \circ \varepsilon_i \wedge \widetilde{\xi} \circ \varepsilon_i \\ &= \sum_{i=1}^{\ell} \int_F \widetilde{\eta} \circ \varepsilon_i \wedge \rho^*(\varepsilon_i) \widetilde{\xi} = \sum_{i=1}^{\ell} \int_F \rho(\varepsilon_i)^{-1} \widetilde{\eta} \circ \varepsilon_i \wedge \widetilde{\xi}.\end{aligned}$$

Thus, if Φ_η denotes the r-form on $\mathcal{D}$ defined by

$$\Phi_\eta = \sum_{i=1}^{\ell} \rho(\varepsilon_i)^{-1} \varepsilon_i^* \widetilde{\eta} = \sum_{i=1}^{\ell} \rho(\varepsilon_i)^{-1} \widetilde{\eta} \circ \varepsilon_i \in \mathcal{E}^r(\mathcal{D}), \tag{5.12}$$

then we have

$$\int_{X'} \eta \wedge \mu^*(\xi) = \int_F \Phi_\eta \wedge \widetilde{\xi}.$$

We now need to show that Φ_η is an element of $\mathcal{E}^r(\mathcal{D}, \Gamma, \rho)$. Given $\gamma \in \Gamma$ and a positive integer $i \leq \ell$, using (5.11), we see that

$$\Gamma' \varepsilon_i \gamma = \Gamma' \varepsilon_{i_\gamma}$$

for some $i_\gamma \in \{1, \dots, \ell\}$. Thus there is an element $\delta_i \in \Gamma'$ such that

$$\varepsilon_i \gamma = \delta_i \varepsilon_{i_\gamma}.$$

Using this and (5.12), we see that

$$\begin{aligned}\Phi_\eta \circ \gamma &= \sum_{i=1}^{\ell} \rho(\varepsilon_i)^{-1}(\widetilde{\eta} \circ \varepsilon_i) \circ \gamma = \sum_{i=1}^{\ell} \rho(\varepsilon_i)^{-1}\widetilde{\eta} \circ (\varepsilon_i \gamma)\\ &= \sum_{i=1}^{\ell} \rho(\varepsilon_i)^{-1}\widetilde{\eta} \circ (\delta_i \varepsilon_{i_\gamma}) = \sum_{i=1}^{\ell} \rho(\varepsilon_i)^{-1}(\widetilde{\eta} \circ \delta_i) \circ \varepsilon_{i_\gamma}\\ &= \sum_{i=1}^{\ell} \rho(\varepsilon_i^{-1}\delta_i)\widetilde{\eta} \circ \varepsilon_{i_\gamma} = \sum_{i=1}^{\ell} \rho(\gamma \varepsilon_{i_\gamma}^{-1})\widetilde{\eta} \circ \varepsilon_{i_\gamma}\\ &= \rho(\gamma) \sum_{i=1}^{\ell} \rho(\varepsilon_{i_\gamma}^{-1})\widetilde{\eta} \circ \varepsilon_{i_\gamma} = \rho(\gamma)\Phi_\eta,\end{aligned}$$

which shows that Φ_η belongs to $\mathcal{E}^r(\mathcal{D}, \Gamma, \rho)$. Hence it follows that $\mu_!(\eta)$ is the element of $\mathcal{E}^r(X, \mathcal{V}_\rho)$ corresponding to $\Phi_\eta \in \mathcal{E}^r(\mathcal{D}, \Gamma, \rho)$ under the canonical isomorphism (4.17). Therefore we obtain

$$T^r(\tau, \mu)\omega = \sum_{i=1}^{\ell} \rho(\varepsilon_i)^{-1}\varepsilon_i^*(\tau^*\omega) = \sum_{i=1}^{\ell} \rho(\varepsilon_i)^{-1}(\tau \circ \varepsilon_i)^*\omega$$

for all $\omega \in \mathcal{E}^r(X, \mathcal{V}_\rho)$, and hence the proof of the theorem is complete. □

6. Compatibility of Hecke operators

The goal of this section is to establish the compatibility among the Hecke operators acting on various types of cohomology groups. Given a discrete group Γ acting on a Riemannian symmetric space $\mathcal{D}$ and a representation $\rho : \Gamma \to GL(F)$ of Γ in a finite-dimensional vector space F, Section 6.1 describes the canonical isomorphisms among the de Rham cohomology of $\mathcal{D}$ associated to ρ, the cohomology of Γ with coefficients in F, and the equivariant C^∞ singular cohomology of $\mathcal{D}$. The compatibility between Hecke operators on singular cohomology and the ones on de Rham cohomology is discussed in Section 6.2. In Section 6.3 it is shown that the Hecke operators on the de Rham cohomology and those on the group cohomology are compatible under the canonical isomorphism obtained in Section 6.1.

6.1. **De Rham, singular and group cohomology.** Let $\mathcal{D}$ be a Riemannian symmetric space, and let Γ be a discrete group acting on $\mathcal{D}$ properly discontinuously. We regard the associated quotient space $X = \Gamma \backslash \mathcal{D}$ as a subset of $\mathcal{D}$ consisting of the set of representatives of the orbits of Γ in $\mathcal{D}$. We shall review relations between singular and group cohomology discussed by Eilenberg in [4].

Let $\Xi : s \to \mathcal{D}$ be a singular q-simplex in $\mathcal{D}$, where $s = \langle p_0, \dots, p_q \rangle$ is a Euclidean simplex with ordered vertices $p_0 < \cdots < p_q$. Then the vertices $\Xi(p_0), \dots, \Xi(p_q)$ of Ξ in $\mathcal{D}$ can be written uniquely in the form

$$\Xi(p_0) = \delta_0 x_0, \dots, \Xi(p_q) = \delta_q x_q \tag{6.1}$$

for some $x_0, \dots, x_q \in X$. Let $K_{\mathcal{D}}$ be the singular complex in $\mathcal{D}$ as in Section 5.2, and let K_Γ be the complex for the cohomology of the group Γ considered in Section 3.2. If Ξ is as in (6.1), we define $\tau_q \Xi$ to be the q-cell of K_Γ given by

$$\tau_q \Xi = \langle \delta_0, \dots, \delta_q \rangle.$$

Thus τ_q maps the singular simplexes in $\mathcal{D}$ into cells of K_Γ and induces a homomorphism

$$\tau_q : C_q(\mathcal{D}) = C_q(K_{\mathcal{D}}) \to C_q(K_\Gamma) \tag{6.2}$$

of groups of q-chains. We also see that

$$\partial_\Gamma \circ \tau_q = \tau_{q-1} \circ \partial_{\mathcal{D}} \tag{6.3}$$

for all $q \geq 1$.

Definition 6.1. (i) A q-cell c_q in K_Γ is said to be *basic* if its first vertex is the identity element of Γ, that is, if $c_q = (1, \gamma_1, \dots, \gamma_q)$ for some $\gamma_1, \dots, \gamma_q \in \Gamma$.

(ii) A simplex Ξ in $\mathcal{D}$ is called *basic* if its leading vertex is one of the points in X, where X is regarded as a subset of $\mathcal{D}$ consisting of the set of representatives of Γ-orbits in $\mathcal{D}$ as above.

Lemma 6.2. *If $n = \dim \mathcal{D}$, then for each integer q with $0 \leq q \leq n$ there is a homomorphism*

$$\nu_q : C_q(K_\Gamma) \to C_q(\mathcal{D}) \tag{6.4}$$

satisfying

$$\partial_{\mathcal{D}} \circ \nu_q = \nu_{q-1} \circ \partial_\Gamma, \quad \gamma \nu_q = \nu_q \gamma \tag{6.5}$$

for all $\gamma \in \Gamma$ with $\nu_{-1} = 0$.

Proof. First, we choose a point $x_0 \in \mathcal{D}$, and define $\nu_0 : C_0(K_\Gamma) \to C_0(\mathcal{D})$ by

$$\nu_0(1) = x_0, \quad \nu_0(\gamma) = \gamma x_0$$

for all $\gamma \in \Gamma$, where (1) is the basic 0-cell. Then we see that

$$\partial_{\mathcal{D}}(\nu_0(\gamma)) = \partial_{\mathcal{D}}(\gamma x_0) = 0 = \nu_{-1}(\partial_\Gamma(\gamma)), \quad \gamma' \nu_0(\gamma) = \gamma' \gamma x_0 = \nu(\gamma' \gamma)$$

for $\gamma, \gamma' \in \Gamma$; hence ν_0 satisfies (6.5). In order to define ν_q for $0 < q \leq n$ by induction, we assume that the maps ν_j have been defined for all j with $0 \leq j < q$ and they satisfy

$$\partial_{\mathcal{D}} \circ \nu_j = \nu_{j-1} \circ \partial_\Gamma, \quad \gamma \nu_j = \nu_j \gamma \tag{6.6}$$

for all $\gamma \in \Gamma$. Given a basic cell σ_q in K_Γ, its image $\nu_{q-1} \partial_\Gamma \sigma_q$ under $\nu_{q-1} \circ \partial_\Gamma$ is an integral chain in $\mathcal{D}$ of dimension $q - 1$, and therefore by the first condition in (6.6) it is a cycle in $\mathcal{D}$, that is,

$$\partial_{\mathcal{D}} \nu_{q-1} \partial_\Gamma \sigma_q = \partial_{\mathcal{D}}^2 \nu_q \sigma_q = 0.$$

Since the space $\mathcal{D}$ is acyclic in dimensions less than n and $q - 1 < n$, there is an element of $C_q(\mathcal{D})$, which we denote by $\nu_q \sigma_q$, such that

$$\partial_{\mathcal{D}} \nu_q \sigma_q = \nu_{q-1} \partial_\Gamma \sigma_q.$$

We now obtain the homomorphism

$$\nu_q : C_q(K_\Gamma) \to C_q(\mathcal{D})$$

by extending the map $\sigma_q \mapsto \nu_q\sigma_q$ to all the q-chains belonging to $C_q(K_\Gamma)$. □

Lemma 6.3. *Given integers q and r with $0 \le q \le n-1$ and $0 \le r \le n$, there exist homomorphisms*

$$P_q : C_q(\mathcal{D}) \to C_{q+1}(\mathcal{D}), \quad Q_r : C_r(K_\Gamma) \to C_{r+1}(K_\Gamma)$$

satisfying the relations

$$\gamma P_q = P_q\gamma, \quad \partial_{\mathcal{D}} P_q c = c - \nu_q\tau_q c - P_q\partial_{\mathcal{D}} c, \tag{6.7}$$

$$\gamma Q_r = Q_r\gamma, \quad \partial_\Gamma Q_r c' = c' - \tau_q\nu_q c' - Q_r\partial_\Gamma c' \tag{6.8}$$

for all $c \in C_q(\mathcal{D})$, $c' \in C_q(K_\Gamma)$ and $\gamma \in \Gamma$.

Proof. Given a basic 0-cell x in $\mathcal{D}$, we consider the 0-cycle $x - \nu_q\tau_q x$. Since $\nu_q\tau_q x = \nu_q(1)$ is a point in $\mathcal{D}$ and since $\mathcal{D}$ is pathwise connected, there is an integral 1-chain $P_0 x$ such that $\partial_{\mathcal{D}} P_0 x = x - \nu_q\tau_q x$. We extend this to the nonbasic 0-cells by

$$P_0(\gamma x) = \gamma P_0(x)$$

for all $\gamma \in \Gamma$, and use induction for general P_q as follows. Assume that P_j has been defined for all j-cells with $j < q < n$ and that the relations in (6.7) hold. Given a basic q-simplex η of $\mathcal{D}$, consider the q-chain

$$c = \eta - \nu_q\tau_q\eta - P_{q-1}\partial_{\mathcal{D}}\eta$$

in $\mathcal{D}$. Then we have

$$\begin{aligned}\partial_{\mathcal{D}} c &= \partial_{\mathcal{D}}\eta - \partial_{\mathcal{D}}\nu_q\tau_q\eta - \partial_{\mathcal{D}} P_{q-1}\partial\eta \\ &= \partial_{\mathcal{D}}\eta - \nu_q\tau_q\partial_{\mathcal{D}}\eta - (\partial_{\mathcal{D}}\eta - \nu_q\tau_q\partial_{\mathcal{D}}\eta - P_{q-1}\partial_{\mathcal{D}}^2\eta) = 0,\end{aligned}$$

and hence the chain c is an q-cycle in $\mathcal{D}$. Since $H_q(\mathcal{D}) = 0$, there is an $(q+1)$-chain $P_q\eta$ such that $\partial_{\mathcal{D}} P_q\eta = c$. We now extend the map P_q to all the q-chains in $C_q(\mathcal{D})$ including nonbasic ones by using the first condition in (6.7). The construction of Q can be obtained in a similar manner. □

Theorem 6.4. *Let $\mathcal{D}$ be a topological space that is acyclic in dimensions less than n, and let Γ be a group acting on $\mathcal{D}$ without fixed points. If A is a left Γ-module, then we have*

$$H^q(\Gamma, A) \cong H^q_E(\mathcal{D}, A) \tag{6.9}$$

for $1 \le q \le n-1$, and

$$H^n(\Gamma, A) \cong \operatorname{Ker}[\phi : H^n_E(\mathcal{D}, A) \to H^n(\mathcal{D}, A)], \tag{6.10}$$

where the homomorphism ϕ is from the exact sequence (3.14) for the complex $K_{\mathcal{D}}$.

Proof. Since the map $\tau_n : C_n(\mathcal{D}) \to C_n(K_\Gamma)$ in (6.2) satisfies (6.3), it induces the homomorphism

$$\tau^q_E : H^q_E(K_\Gamma, A) \to H^q_E(\mathcal{D}, A) \tag{6.11}$$

for each q. From (6.7) and (6.8) we see that the maps $\nu_q\tau_q$ and $\tau_q\nu_q$ are chain homotopic to the identity maps $1 : C_q(\mathcal{D}) \to C_q(\mathcal{D})$ and $1 : C_q(K_\Gamma) \to C_q(K_\Gamma)$, respectively. Since the maps ν_q, P_q and Q_q are equivariant, for $q < n$ the homomorphisms in (6.11) are isomorphisms. Hence we obtain (6.9) by combining the isomorphism τ_E^q with the relation (3.17). In order to prove (6.10) we consider the commutative diagram

$$\begin{array}{ccccccc} H_{n-1}(\mathcal{D},A) & \xrightarrow{\psi_{\mathcal{D}}} & H_R^{n-1}(\mathcal{D},A) & \xrightarrow{\delta_{\mathcal{D}}} & H_E^n(\mathcal{D},A) & \xrightarrow{\phi_{\mathcal{D}}} & H^n(\mathcal{D},A) \\ \downarrow & & \tau_R^{n-1}\downarrow & & \tau_E^n\downarrow & & \downarrow \\ H_{n-1}(K_\Gamma,A) & \xrightarrow{\psi_\Gamma} & H_R^{n-1}(K_\Gamma,A) & \xrightarrow{\delta_\Gamma} & H_E^n(K_\Gamma,A) & \xrightarrow{\phi_\Gamma} & H^n(K_\Gamma,A) \end{array}$$

induced by τ and the exact sequence (3.14). Since $H_{n-1}(K_\Gamma,A) = H_n(K_\Gamma,A) = 0$ (see [4, p. 47]), the map δ_Γ is an isomorphism. On the other hand, δ is injective because $H^{n-1}(K_\Gamma,A) = 0$. Using the relation $\delta_\Gamma\tau_E^n = \tau_R^{n-1}\delta$ and the fact that both τ_R^{n-1} and δ_Γ are isomorphisms, we see that τ_R^n is injective and has the same image as δ. However, we have

$$\delta H_R^{n-1}(\mathcal{D},A) = \operatorname{Ker}\phi = K_E^n(\mathcal{D},A);$$

hence we obtain (6.10). □

Let $\rho : \Gamma \to GL(F)$ be a representation of Γ in a finite-dimensional vector space F over $\mathbb{R}$ as in Section 5.3, so that F can be regarded as a left Γ-module.

Proposition 6.5. *There is a canonical isomorphism*

$$H^q(\mathcal{D},\Gamma,\rho) \cong H^q_{\infty,E}(\mathcal{D},F) \tag{6.12}$$

for each $q \geq 0$.

Proof. If f_ω with $\omega \in \mathcal{E}^p(\mathcal{D},F)$ is as in (4.15), then by Theorem 4.8 the map $\omega \mapsto f_\omega$ determines an isomorphism between $H^q_{\mathrm{DR}}(\mathcal{D},F)$ and $H^q_\infty(\mathcal{D},F)$. On the other hand, if $\omega \in \mathcal{E}^q(\mathcal{D},\Gamma,\rho)$, we have

$$f_\omega(\gamma c) = \int_{\gamma c}\omega = \int_c \gamma^*\omega = \int_c \omega\circ\gamma = \int_c \rho(\gamma)\omega = \rho(\gamma)\int_c\omega = \rho(\gamma)f_\omega(c)$$

for all $\gamma \in \Gamma$ and $c \in \mathcal{S}_q^\infty$. Thus it follows that f_ω is equivariant, and therefore the map $\omega \mapsto f_\omega$ determines an isomorphism (6.12). □

Corollary 6.6. *If $\mathcal{D}$ is contractible, there is a canonical isomorphism*

$$H^q(\mathcal{D},\Gamma,\rho) \cong H^q(\Gamma,F)$$

for each $q \geq 0$, where F is regarded as a Γ-module via the representation ρ.

Proof. This follows from the isomorphisms (6.9) and (6.12). □

6.2. **Singular and de Rham cohomology.** Let G be a semisimple Lie group, and let $\mathcal{D}$ be the associated symmetric space, which can be identified with the quotient G/K of G by a maximal compact subgroup. Let Γ be a discrete subgroup of G, and let $X = \Gamma\backslash\mathcal{D}$ be the associated locally symmetric space. Let $\rho : G \to GL(F)$ be a representation of G in a finite-dimensional real vector space F.

Given $q \geq 0$, the group $C^q_\infty(\mathcal{D},F)$ of C^∞ q-cochains in $\mathcal{D}$ with coefficients in F can be written as

$$C^q_\infty(\mathcal{D},F) = \operatorname{Hom}(C_q^\infty,F),$$

where C_q^∞ denotes the group of C^∞ q-chains in $\mathcal{D}$. If $\alpha \in \widetilde{\Gamma}$ with

$$\Gamma\alpha\Gamma = \coprod_{i=1}^{d} \Gamma\alpha_i, \tag{6.13}$$

we define the map $T_s(\alpha) : C_\infty^q(\mathcal{D}, F) \to C_\infty^q(\mathcal{D}, F)$ by

$$(T_s(\alpha)f)(c) = \sum_{i=1}^{d} \rho(\alpha_i)^{-1} f(\alpha_i c) \tag{6.14}$$

for all $f \in C_\infty^q(\mathcal{D}, F)$ and $c \in C_q^\infty(\mathcal{D})$. Since clearly $T_s(\alpha)$ commutes with the boundary operator for the complex $C_\infty^\bullet(\mathcal{D}, F)$, it induces the map

$$T_s(\alpha) : H_\infty^q(\mathcal{D}, F) \to H_\infty^q(\mathcal{D}, F), \tag{6.15}$$

which is the *Hecke operator* on the q-th C^∞ singular cohomology group $H_\infty^q(\mathcal{D}, F)$ with coefficients in F.

Lemma 6.7. *The map $T_s(\alpha) : C_\infty^q(\mathcal{D}, F) \to C_\infty^q(\mathcal{D}, F)$ given by (6.14) sends Γ-equivariant q-cochains to Γ-equivariant q-cochains.*

Proof. Let α be an element of $\widetilde{\Gamma}$ such that the corresponding double coset has a decomposition given by (6.13). Then, as in (3.21), for each $i \in \{1, \dots, d\}$ and $\gamma \in \Gamma$ there are elements $i(\gamma) \in \{1, \dots, d\}$ and $\xi_i(\gamma) \in \Gamma$ such that

$$\alpha_i \gamma = \xi_i(\gamma) \cdot \alpha_{i(\gamma)}. \tag{6.16}$$

Furthermore, the set $\{\alpha_{1(\gamma)}, \dots, \alpha_{d(\gamma)}\}$ is a permutation of $\{\alpha_1, \dots, \alpha_d\}$ for each $\gamma \in \Gamma$. Let $f \in C_{E,\infty}^q(\mathcal{D}, F)$ and $c \in C_q^\infty(\mathcal{D})$, where $C_{E,\infty}^q(\mathcal{D}, F)$ is the subspace of $C_\infty^q(\mathcal{D}, F)$ consisting of Γ-equivariant cochains. Then, since f is Γ-equivariant, we have

$$f(\delta c) = \rho(\delta) f(c)$$

for all $\delta \in \Gamma$. Using this, (6.14) and (6.16), we obtain

$$\begin{aligned}
(T_s(\alpha)f)(\gamma c) &= \sum_{i=1}^{d} \rho(\alpha_i)^{-1} f(\alpha_i \gamma c) \\
&= \sum_{i=1}^{d} \rho(\xi_i(\gamma)\alpha_{i(\gamma)}\gamma^{-1})^{-1} f(\xi_i(\gamma)\alpha_{i(\gamma)} c) \\
&= \sum_{i=1}^{d} \rho(\gamma)\rho(\alpha_{i(\gamma)})^{-1} \rho(\xi_i(\gamma))^{-1} \rho(\xi_i(\gamma)) f(\alpha_{i(\gamma)} c) \\
&= \rho(\gamma) \sum_{i=1}^{d} \rho(\alpha_{i(\gamma)})^{-1} f(\alpha_{i(\gamma)} \gamma c) = \rho(\gamma)(T(\alpha)f)(c)
\end{aligned}$$

for all $\gamma \in \Gamma$. Thus it follows that $T(\alpha)f \in C_{E,\infty}^q(\mathcal{D}, F)$. □

Given $\alpha \in \widetilde{\Gamma}$, we denote by $\Gamma_{(\alpha)}$ the subgroup of Γ defined by

$$\Gamma_{(\alpha)} = \Gamma \cap \alpha^{-1}\Gamma\alpha, \tag{6.17}$$

and set

$$X_{(\alpha)} = \Gamma_{(\alpha)} \backslash \mathcal{D}.$$

We assume that $[\Gamma : \Gamma_{(\alpha)}] = d$ and that

$$\Gamma = \coprod_{i=1}^{d} \Gamma_{(\alpha)} \delta_i \tag{6.18}$$

with $\delta_1, \ldots, \delta_s \in \Gamma$. Then by Lemma 2.4 we have

$$\Gamma\alpha\Gamma = \coprod_{i=1}^{d} \Gamma\alpha\delta_i. \tag{6.19}$$

If $\gamma' \in \Gamma_{(\alpha)}$, then $\alpha\gamma'\alpha^{-1} \in \Gamma$; hence for each $z \in \mathcal{D}$ we have

$$\alpha(\gamma' z) = (\alpha\gamma'\alpha^{-1})\alpha z \in \Gamma(\alpha z).$$

Thus it follows that $\alpha(\Gamma_{(\alpha)} z) \subset \Gamma(\alpha z)$, and therefore the map $\alpha : \mathcal{D} \to \mathcal{D}$, $z \mapsto \alpha z$ induces a map $\mu_\alpha : X_{(\alpha)} \to X$. However, since $\Gamma_{(\alpha)} \subset \Gamma$, there is another map $\mu_1 : X_{(\alpha)} \to X$ induced by the identity map on $\mathcal{D}$. Thus the maps μ_1 and μ_α are d-sheeted covering maps of X, and by Definition 5.6 they determine the Hecke operator $T^r(\mu_1, \mu_\alpha)$ on $H^r(X, \mathcal{V}_\rho)$ for each $r \geq 0$. By identifying $H^r(X, \mathcal{V}_\rho)$ with $H^r(\mathcal{D}, \Gamma, \rho)$ using the canonical isomorphism (4.17) we obtain the Hecke operator

$$T^r(\mu_1, \mu_\alpha) : H^r(\mathcal{D}, \Gamma, \rho) \to H^r(\mathcal{D}, \Gamma, \rho)$$

for each $r \geq 0$. On the other hand, by Lemma 6.7 the Hecke operator (6.15) induces the Hecke operator

$$T_s(\alpha) : H^q_{E,\infty}(\mathcal{D}, F) \to H^q_{E,\infty}(\mathcal{D}, F)$$

on the C^∞ q-th equivariant cohomology group with coefficients in F. We denote by

$$\phi : H^q(\mathcal{D}, \Gamma, \rho) \to H^q_{E,\infty}(\mathcal{D}, F)$$

the canonical isomorphism (6.12).

Theorem 6.8. *Given $\alpha \in \widetilde{\Gamma}$ and $q \geq 0$, we have*

$$\phi(T^q(\mu_1, \mu_\alpha)[\omega]) = T_s(\alpha)\phi([\omega])$$

for all $\omega \in C^q(\mathcal{D}, \Gamma, \rho)$, where $[\omega]$ denotes the cohomology class of ω in $H^q(\mathcal{D}, \Gamma, \rho)$.

Proof. Let $\omega \in C^q(\mathcal{D}, \Gamma, \rho)$, and let α be an element of $\widetilde{\Gamma}$ satisfying (6.13) and (6.19). Then, using (6.14) and Theorem 5.7, we have

$$\begin{aligned}\phi(T^q(\mu_1, \mu_\alpha)[\omega])(c) &= \int_c T^q(\mu_1, \mu_\alpha)\omega = \sum_{i=1}^d \int_c \rho(\alpha\delta_i)^{-1}\omega \circ (\alpha\delta_i) \\ &= \sum_{i=1}^d \int_{\alpha\delta_i c} \rho(\alpha\delta_i)^{-1}\omega = \sum_{i=1}^d \rho(\alpha\delta_i)^{-1}\phi(\omega)(\alpha\delta_i c) \\ &= (T_s(\alpha)\phi([\omega]))(c)\end{aligned}$$

for all $c \in C_q^\infty(\mathcal{D})$; hence the theorem follows. □

6.3. **De Rham and group cohomology.** Let G, K, $\mathcal{D} = G/K$ and the representation $\rho : \Gamma \to GL(F)$ of a discrete subgroup Γ of G in F be as in Section 6.2.

Let $C_q(\mathcal{D})$ be the group of singular q-chains as in Section 5.2, and let $\Xi : s \to \mathcal{D}$ be a singular q-simplex belonging to $C_q(\mathcal{D})$, where $s = \langle p_0, \dots, p_q\rangle$ is a Euclidean simplex with ordered vertices $p_0 < \cdots < p_q$. Then, as in Section 6.1, the vertices $\Xi(p_0), \dots, \Xi(p_q)$ of Ξ can be written uniquely in the form

$$\Xi(p_0) = \delta_0 x_0, \dots, \Xi(p_q) = \delta_q x_q$$

for some $x_0, \dots, x_q \in X$, where $X = \Gamma\backslash\mathcal{D}$ is regarded as a subset of $\mathcal{D}$ consisting of representatives of the orbits of Γ. Given a q-form $\omega \in \mathcal{E}^q(\mathcal{D})$ on $\mathcal{D}$, we define the associated map $\mathcal{F}(\omega) : \Gamma^{q+1} \to F$ by

$$\mathcal{F}(\omega)(\gamma_0, \dots, \gamma_q) = \int_{\nu_q(\gamma_0,\dots,\gamma_q)} \omega \tag{6.20}$$

for all $\gamma_0, \dots, \gamma_q \in \Gamma$, where $\nu_q : C_q(K_\Gamma) \to C_q(\mathcal{D})$ is as in (6.4).

Lemma 6.9. *If $\omega \in \mathcal{E}^q(\mathcal{D}, \Gamma, \rho)$, then we have*

$$\mathcal{F}(\omega)(\gamma\gamma_0, \dots, \gamma\gamma_q) = \rho(\gamma)^{-1}\mathcal{F}(\omega)(\gamma_0, \dots, \gamma_q) \tag{6.21}$$

for all $\gamma, \gamma_0, \dots, \gamma_q \in \Gamma$.

Proof. Given $\gamma_0, \dots, \gamma_q \in \Gamma$, using the construction of ν_q in the proof of Lemma (4.8), we see easily that

$$\nu_q(\gamma\gamma_0, \dots, \gamma\gamma_q) = \gamma\nu_q(\gamma_0, \dots, \gamma_q)$$

for all $\gamma \in \Gamma$. Hence by using (4.16) we obtain

$$\begin{aligned}\mathcal{F}(\omega)(\gamma\gamma_0, \dots, \gamma\gamma_q) &= \int_{\gamma\nu_q(\gamma_0,\dots,\gamma_q)} \omega = \int_{\nu_q(\gamma_0,\dots,\gamma_q)} \omega \circ \gamma \\ &= \int_{\nu_q(\gamma_0,\dots,\gamma_q)} \rho(\gamma)^{-1}\omega \\ &= \rho(\gamma)^{-1}\mathcal{F}(\omega)(\gamma_0, \dots, \gamma_q)\end{aligned}$$

for all $\omega \in \mathcal{E}^q(\mathcal{D}, \Gamma, \rho)$. □

By Lemma 6.9 the map $\mathcal{F}(\omega) : \Gamma^{q+1} \to F$ given by (6.20) is a homogeneous q-cochain for the cohomology of Γ described in Section 3.1. Thus we have

$$\mathcal{F}(\omega) \in \mathfrak{C}^q(\Gamma, F)$$

for all $\omega \in \mathcal{E}^q(\mathcal{D}, \Gamma, \rho)$, where F is regarded as a Γ-module via the representation ρ. We denote by δ and ∂ the coboundary maps for the complexes $\mathcal{E}^\bullet(\mathcal{D}, \Gamma, \rho)$ and $\mathfrak{C}^\bullet(\Gamma, F)$, respectively.

Lemma 6.10. *The map $\omega \mapsto \mathcal{F}(\omega)$ given by* (6.20) *satisfies*

$$\delta\mathcal{F}(\omega) = \mathcal{F}(d\omega)$$

for all $\omega \in \mathcal{E}^q(\mathcal{D}, \Gamma, \rho)$.

Proof. Given $\gamma_0, \dots, \gamma_{q+1} \in \Gamma$ and $\omega \in \mathcal{E}^q(\mathcal{D}, \Gamma, \rho)$, using (6.20), we see that

$$\begin{aligned}\mathcal{F}(d\omega)(\gamma_0, \dots, \gamma_{q+1}) &= \int_{\nu_{q+1}(\gamma_0,\dots,\gamma_{q+1})} d\omega = \int_{\partial_{\mathcal{D}}\nu_{q+1}(\gamma_0,\dots,\gamma_{q+1})} \omega \\ &= \int_{\nu_q \partial_\Gamma(\gamma_0,\dots,\gamma_{q+1})} \omega = \mathcal{F}(\omega)(\partial_\Gamma(\gamma_0, \dots, \gamma_{q+1})),\end{aligned}$$

where we used the relation $\partial_{\mathcal{D}} \circ \nu_{q+1} = \nu_q \circ \partial_\Gamma$ from (6.5). However, since the boundary operator ∂_Γ and the coboundary operator ∂_Γ are given by (3.15) and (3.2), respectively, we have

$$\begin{aligned}\mathcal{F}(\omega)(\partial_\Gamma(\gamma_0, \dots, \gamma_{q+1})) &= \sum_{i=0}^{q+1} (-1)^i \mathcal{F}(\omega)(\gamma_0, \dots, \widehat{\gamma}_i, \dots, \gamma_{q+1}) \\ &= (\delta\mathcal{F}(\omega))(\gamma_0, \dots, \gamma_{q+1});\end{aligned}$$

hence the lemma follows. □

By Lemma 6.10 the map $\mathcal{F} : \mathcal{E}^q(\mathcal{D}, \Gamma, \rho) \to \mathfrak{C}^q(\Gamma, F)$ given by (6.20) induces the canonical isomorphism

$$\mathcal{F} : H^q(\mathcal{D}, \Gamma, \rho) \to H^q(\Gamma, F) \tag{6.22}$$

for each $q \geq 0$.

Lemma 6.11. *Let $(\gamma_0, \dots, \gamma_q) \in C_q(K_\Gamma)$, and let $c \in C_q(\mathcal{D})$ be a q-cycle such that $\tau_q(c) = (\gamma_0, \dots, \gamma_q)$, where $\tau_q : C_q(\mathcal{D}) \to C_1(K_\Gamma)$ is as in* (6.2)*. Then we have*

$$\mathcal{F}(\omega)(\gamma_0, \dots, \gamma_q) = \int_c \omega$$

for all closed q-forms $\omega \in \mathcal{E}^q(\mathcal{D}$.

Proof. If $\tau_q(c) = (\gamma_0, \dots, \gamma_q)$, by using (6.7) we see that

$$\nu_q(\gamma_0, \dots, \gamma_q) = \nu_q \tau_q c = c - P_q \partial_{\mathcal{D}} c - \partial_{\mathcal{D}} P_q c = c - \partial_{\mathcal{D}} P_q c,$$

where we used the fact that c is a cycle. Thus the formula (6.20) can be written in the form

$$\mathcal{F}(\omega)(\gamma_0, \dots, \gamma_q) = \int_{c - \partial_{\mathcal{D}} P_q c} \omega = \int_c \omega - \int_{P_q c} d\omega = \int_c \omega,$$

since ω is a closed form. □

Let G be a reductive group containing Γ, and let $\widetilde{\Gamma} \subset G$ be the commensurability group of Γ. Given $\alpha \in \widetilde{\Gamma}$ and $q \geq 0$, let $\mathfrak{T}(\alpha) : H^q(\Gamma, F) \to H^q(\Gamma, F)$ be the Hecke operator on group cohomology described in Section 3.3. Let $T(\alpha) : H^q(\mathcal{D}, \Gamma, \rho) \to H^q(\mathcal{D}, \Gamma, \rho)$ be the Hecke operator in (4.21), which may be regarded as a Hecke operator on $H^q(X, \mathcal{V}_\rho)$ by using the canonical isomorphism

$$H^q(X, \mathcal{V}_\rho) \cong H^q(\mathcal{D}, \Gamma, \rho)$$

considered in (4.17).

Theorem 6.12. *Let $\mathcal{F}$ be the isomorphism in (6.22). Then we have*

$$\mathfrak{T}(\alpha) \circ \mathcal{F} = \mathcal{F} \circ T(\alpha)$$

for all $\alpha \in \widetilde{\Gamma}$.

Proof. Assume that the double coset containing $\alpha \in \widetilde{\Gamma}$ has a decomposition of the form

$$\Gamma\alpha\Gamma = \coprod_{1 \leq i \leq d} \Gamma\alpha_i$$

for some elements $\alpha_1, \ldots, \alpha_d \in \widetilde{\Gamma}$. If $\gamma \in \Gamma$ and $1 \leq i \leq d$, as was described in (3.21), we have

$$\alpha_i \gamma = \xi_i(\gamma) \cdot \alpha_{i(\gamma)}$$

for some element $\xi_i(\gamma) \in \Gamma$, where $\{\alpha_{1(\gamma)}, \ldots, \alpha_{d(\gamma)}\}$ is a permutation of $\{\alpha_1, \ldots, \alpha_d\}$. Since F is a Γ-module via the representation ρ, the formula (3.24) can be written in the form

$$(\mathfrak{T}(\alpha)f)(\gamma_0, \ldots, \gamma_q) = \sum_{i=1}^{d} \rho(\alpha_i)^{-1} f(\xi_i(\gamma_0), \ldots, \xi_i(\gamma_q))$$

for each q-cocycle $f \in \mathfrak{Z}(\Gamma, F)$ and $\gamma_0, \ldots, \gamma_q \in \Gamma$. Thus we have

$$(\mathfrak{T}(\alpha)\mathcal{F}(\omega))(\gamma_0, \ldots, \gamma_q) = \sum_{i=1}^{d} \rho(\alpha_i)^{-1} \mathcal{F}(\omega)(\xi_i(\gamma_0), \ldots, \xi_i(\gamma_q)) \tag{6.23}$$

for all $\omega \in \mathcal{E}^q(\mathcal{D}, \Gamma, \rho)$. We now fix a point $x_0 \in \mathcal{D}$ and choose the set X of representatives of Γ-orbits in such a way that

$$x_0, \alpha_1^{-1} x_0, \ldots, \alpha_d^{-1} x_0 \in X. \tag{6.24}$$

Then, if $\tau_q : C_q(\mathcal{D}) \to C_1(K_\Gamma)$ is as in (6.2), we see that

$$\tau_q \langle \xi_i(\gamma_0) x_0, \ldots, \xi_i(\gamma_q) x_0 \rangle = (\xi_i(\gamma_0), \ldots, \xi_i(\gamma_q))$$

for $1 \leq i \leq d$. From this and Lemma 6.11, we obtain

$$\mathcal{F}(\omega)(\xi_i(\gamma_0), \ldots, \xi_i(\gamma_q)) = \int_{\langle \xi(\gamma_0) x_0, \ldots, \xi(\gamma_q) x_0 \rangle} \omega.$$

Using this, (6.23), and the relation $\xi_i(\gamma) = \alpha_i \gamma \alpha_{i(\gamma)}^{-1}$ for $1 \leq i \leq d$, we have

$$\begin{aligned}(\mathfrak{T}(\alpha)\mathcal{F}(\omega))(\gamma_0, \dots, \gamma_q) &= \sum_{i=1}^{d} \rho(\alpha_i)^{-1} \int_{\alpha_i \langle \gamma_0 \widetilde{x}_0, \dots, \gamma_q \widetilde{x}_q \rangle} \omega \\ &= \sum_{i=1}^{d} \int_{\langle \gamma_0 \widetilde{x}_0, \dots, \gamma_q \widetilde{x}_q \rangle} \rho(\alpha_i)^{-1} \omega \circ \alpha_i\end{aligned}$$

where $\widetilde{x}_j = \alpha_{i(\gamma_j)}^{-1} x$ for each j. However, since $\{\alpha_{1(\gamma_j)}, \dots, \alpha_{d(\gamma_j)}\}$ is a permutation of $\{\alpha_1, \dots, \alpha_d\}$, the condition (6.24) implies that $\widetilde{x}_j \in X$ for each $j \in \{0, 1, \dots, q\}$. Hence we have

$$\langle \gamma_0 \widetilde{x}_0, \dots, \gamma_q \widetilde{x}_q \rangle = \tau_q(\gamma_0, \dots, \gamma_q),$$

and therefore it follows that

$$\begin{aligned}(\mathfrak{T}(\alpha)\mathcal{F}(\omega))(\gamma_0, \dots, \gamma_q) &= \sum_{i=1}^{d} (\mathcal{F}(\rho(\alpha_i)^{-1} \omega \circ \alpha_i))(\gamma_0, \dots, \gamma_q) \\ &= (\mathcal{F}(T(\alpha)\omega))(\gamma_0, \dots, \gamma_q).\end{aligned}$$

Thus we obtain $\mathfrak{T}(\alpha) \circ \mathcal{F} = \mathcal{F} \circ T(\alpha)$, and the proof of the theorem is complete. □

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